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Optimal approximation of piecewise smooth functions using deep ReLU neural networks

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Abstract

We study the necessary and sufficient complexity of ReLU neural networks—in terms of depth and number of weights—which is required for approximating classifier functions in an L^2-sense.

As a model class, we consider the set $\mathcal{E}^\beta(\mathbb{R}^d)$ of possibly discontinuous piecewise C^β functions $f : [-1/2, 1/2]^d \to \mathbb{R}$, where the different "smooth regions" of f are separated by C^β hypersurfaces. For given dimension $d \geq 2$, regularity $\beta > 0$, and accuracy $\varepsilon > 0$, we construct artificial neural networks with ReLU activation function that approximate functions from $\mathcal{E}^\beta(\mathbb{R}^d)$ up to an L^2 error of ε. The constructed networks have a fixed number of layers, depending only on d and β and they have $\mathcal{O}(\varepsilon^{-2(d-1)/\beta})$ many non-zero weights, which we prove to be optimal. For the proof of optimality, we establish a lower bound on the description complexity of the class $\mathcal{E}^\beta(\mathbb{R}^d)$. By showing that a family of approximating neural networks gives rise to an encoder for $\mathcal{E}^\beta(\mathbb{R}^d)$, we then prove that one cannot approximate a general function $f \in \mathcal{E}^\beta(\mathbb{R}^d)$ using neural networks that are less complex than those produced by our construction.

In addition to the optimality in terms of the number of weights, we show that in order to achieve this optimal approximation rate, one needs ReLU networks of a certain minimal depth. Precisely, for piecewise $C^\beta(\mathbb{R}^d)$ functions, this minimal depth is given—up to a multiplicative constant—by β/d. Up to a log factor, our constructed networks match this bound. This partly explains the benefits of depth for ReLU networks by showing that deep networks are necessary to achieve efficient approximation of (piecewise) smooth functions.

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1 Introduction

Neural networks implement functions by connecting multiple simple operations in complex patterns. They were inspired by the architecture of the human brain and in that framework probably first studied in 1943 in [36]. A special network model is that of a *multi-layer perceptron* [44, 43], which can, in mathematical terms, be understood as an alternating concatenation of affine-linear functions and simple nonlinearities, arranged in multiple layers.

Recently, especially deep networks, i.e., those with many layers, have received increased attention, due to the possibility to train them efficiently. In particular, given training data in the form of input and output pairs, there exist highly efficient algorithms to adapt the network in such a way that the network implements an interpolation of the training data and even generalizes well to previously unseen data points, at least for many problems that occur in practice. This procedure is customarily referred to as *deep learning*, [31, 20].

A small selection of spectacular applications of deep learning are image classification [27], speech recognition [24], or game intelligence [15]. While networks trained by deep learning prove to be remarkably versatile and powerful classifiers, it is not entirely understood why these methods work so well. One aspect of the success of deep learning is certainly the powerful network architecture. In mathematical terms, this means that networks yield very efficient approximators for relevant function classes. Note though that this ability to approximate a given function—or to interpolate the training data—does in itself *not* explain why neural networks yield better generalization than other learning architectures. This point, however, is outside the scope of this paper.

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This work extends the study of *approximation properties of neural networks*. We will focus specifically on neural networks that have a certain activation function, which is possibly the most widely used in applications—the rectified linear unit (ReLU). We will determine the optimal trade-off between the complexity and approximation fidelity of neural networks when approximating *piecewise constant or piecewise smooth functions*—function classes that, as we will elaborate upon below, resemble the classifier function of a classification problem. Furthermore, we show that in order to attain this optimal trade-off, one needs networks with a certain minimal depth depending on the smoothness of the functions and on the dimension of their domain.

In the remainder of this introduction, we first motivate our choice of the class of piecewise constant and piecewise smooth functions as functions of interest. Afterwards, we review related results concerning the approximation of (piecewise) smooth functions, both by neural networks and more general function classes. Then, we will clarify our notion of complexity of neural networks. Finally, we describe our contribution and fix some standard and non-standard notation.

1.1 Classification with neural networks

Neural networks are used in a broad range of classification problems: Examples include image classification [27], digit classification [23, 26, 35, 32] or even medical diagnosis [5, 8]. A comprehensive survey on classification by neural networks can be found in [49].

The networks employed in these tasks take very high-dimensional input and assign a simple label to each data point, thereby performing a classification. Thus, we perceive a prototype classifier function as a map $f : \mathbb{R}^d \to \{1, \ldots, N\}$, where N is the number of possible labels. In other words, the *function class of classifier functions is that of piecewise constant functions.* A special case of particular interest is that of binary classification, i.e., where $N = 2$, which is extensively studied in Part 1 of [2].

Admittedly, the model of a classifier function described above is not the only conceivable model. Indeed, another point of view is to consider the classifier function as a conditional probability of the sample having a certain label. In this regard, not piecewise constant functions but rather functions that admit reasonably sharp but smooth phase transitions are the right model.

Which point of view one should adapt naturally depends on the application. To justify our approach, we give one example where a classifier should indeed be piecewise constant. Consider the problem of modeling or predicting some physical behavior. Precisely, let us consider the problem of predicting if a material undergoing some known stress breaks or remains intact. If the underlying physical model is too complicated, it might be reasonable to learn the behavior from data and apply a deep learning approach. In this case, the classifier has two labels—broken and unbroken—and a potentially very high-dimensional input of forces and material properties. Nonetheless, there will be a sharp transition between parameter values that describe stable configurations and those that yield breaks. It is conceivable that one might want to optimize the forces that can be applied, which means that the jump set should be very finely resolved by the learned function.

1.2 Related work on approximation of piecewise smooth functions

We give a short overview of related work on approximation with neural networks and approximation of piecewise smooth functions. In fact, piecewise smooth functions form a superset of the previously described set of piecewise constant functions that describe classifiers, but it will turn out that they admit the same approximation rates with respect to ReLU neural networks. Because of that, it is natural to focus on the larger set of piecewise smooth functions.

One of the central results of approximation with neural networks is the universal approximation theorem [25, 14] stating that every continuous function on a compact domain can be arbitrarily well approximated by a *shallow* neural network, i.e., a network with only one hidden layer. These approximation results only show the possibility of approximation, but do not provide any information on the required size of a network to achieve a given approximation accuracy. Other works analyze the necessary and sufficient size of networks to approximate certain classes of functions, whose Fourier transform has a bounded first moment [4, 3]. In [37], [41] it is shown that *assuming a smooth activation function*, a shallow network with $\varepsilon^{-d/n}$ neurons can uniformly approximate a general C^n-function on a d-dimensional set with infinitesimal error ε. This approximation rate is also demonstrated to be optimal, in the sense that if one insists that the weights of the approximating network should depend *continuously* on the approximated function, the derived rate can not be improved. Note though that in [48, Section 3.3], Yarotsky gives a construction where the weights do *not* depend continuously on the approximated function, and where the "optimal" lower bound is improved by a log factor, but using deep networks instead of shallow ones, and using the ReLU activation function instead of a smooth one. Nonetheless, this result shows that the optimality can indeed fail if the weights are allowed to depend discontinuously on the approximated function.

Except for the result of [48], all the results above concern shallow networks. However, in applications, one observes that *deep networks appear to perform better than shallow ones of comparable size.* Nonetheless, at this point, there does not exist an entirely satisfactory explanation of why this should be the case. Still, from an approximation theoretical point of view, there are a couple of results explaining the connection of depth to the expressive power of a network. In [38] it was demonstrated that deep networks can partition a space into exponentially more linear regions than shallow networks of the same size. [16] analyses special network architectures of sum-product networks and establishes the advantage in the expressive power of deep networks. Moreover, [45] shows the advantage of depth for networks with special piecewise polynomial activation functions. An overview of a large class of functions that can be well approximated with deep but not with shallow networks can be found in [42].

In [48], [46] deep ReLU networks are employed to achieve optimal approximation rates for smooth functions. These results are very closely related to the findings in this paper. However, [48] and [46] consider approximation in the L^∞ norm, which would not be possible for functions with jumps since ReLU networks always implement continuous functions. Finally, we mention [6], where it is demonstrated that for the case of *two-dimensional* piecewise C^α smooth functions with C^α jump curves, $\alpha \in (1, 2]$, neural networks with certain smooth activation functions achieve optimal L^2 approximation. However, these results do not cover the case of networks with a ReLU activation function and do not apply in dimensions $d \neq 2$.

To end this overview of related work, we also give a review on approximation theoretical results for piecewise smooth functions by more general representation systems than neural networks.

Piecewise smooth functions are frequently employed as a model for images in image processing [10, 28, 18], which is why a couple of representation systems developed in that area are particularly well-suited to represent these function classes. For instance, shearlets and curvelets provide optimal N-term approximation rates for piecewise $C^2(\mathbb{R}^2)$ functions with C^2 jump curves, [9, 29, 22, 39].

To obtain optimal approximation of two-dimensional functions with jump curves smoother than C^2, the bandlet system was developed, [40], which is a system consisting of properly smoothly-transformed boundary-adapted wavelets that are optimally adapted to the smooth jump curves.

Another system, the so-called surflets [11], even yields optimal approximation of piecewise smooth functions in $\mathbb{R}^d$. This system is constructed by invoking a partition of unity, as well as local approximation using so-called horizon functions. These ideas are also central to the approximation results in this work.

1.3 Our notion of optimality

To claim that our approximation results are optimal, we need to specify a notion of optimality. First of all, we measure the size of networks mostly in terms of the *number of non-zero weights* in the network. Then we adopt an information theoretical point of view, which was already introduced in [6, 7], but will be refined and improved here. The underlying idea is the following: Under some assumptions on the encodability of the weights of a network, each neural network can be encoded with a bit string the length of which depends only on the number of weights of the network. For a given function class which can be well approximated by neural networks of a given complexity, this gives rise to a lossy compression algorithm for the function class; the error introduced by this compression algorithm depends on the quality of approximation that can be achieved by the given class of networks over the function class. This observation gives rise to an encoding strategy for function classes that are well-approximated by neural networks of limited complexity. In this way, the description complexity of a function class provides a lower bound for the size of the associated networks. Similar ideas associated to lower bounds for the approximation with certain representation systems were used in [17, 21].

Certainly, other means of establishing lower bounds exist. For instance, in [48] known bounds on the Vapnik-Chervonenskis dimension or fat-shattering dimension of networks [2] are used to obtain lower bounds on the achievable approximation rate for a large variety of function classes.

The argument in [48], however, only yields a lower bound regarding the approximation with respect to the L^∞ norm. This is not appropriate in our setting as we study L^2 approximation. Additionally, to obtain *sharp* lower bounds on the approximation using neural networks as in [48], it is necessary to impose an upper bound on the depth of the network. Such an assumption is not required in our approach. On the downside, we require an encodability condition on the weights. A final argument in favor of our optimality criterion is that it is *independent of the chosen activation function* ϱ (as long as $\varrho(0) = 0$), while the arguments in [48, 2] are specific to piecewise polynomial activation functions. A more in-depth comparison of the two approaches is given in Section 4.

A further notion of optimality concerns the number of layers which is necessary to achieve a certain approximation rate by neural networks of that depth. In [42] an overview is given about function classes that can be approximated well by deep networks and which cannot be approximated well with shallow networks. Furthermore, Yarotsky [48] shows that a certain depth is needed to approximate nonlinear C^2 functions with a given approximation rate with respect to the L^∞ norm.

We will discuss this notion in more detail in Section 4. In particular, we will show that the result of Yarotsky can be generalized from L^∞ approximation to approximation in the L^p-sense, for $1 \leq p < \infty$.

1.4 Our contribution

We establish optimal approximation rates for piecewise C^β functions, $\beta \in (0, \infty)$ on $\mathbb{R}^d$, $d \in \mathbb{N}_{\geq 2}$ by ReLU neural networks in terms of the number of non-zero weights and the number of layers. As two special cases, our results cover the approximation of C^β functions and of piecewise constant functions for which the different "constant regions" are separated by hyperplanes of regularity C^β.

A simplified but honest summary of our main results is the following: For a given piecewise C^β function $f : [-1/2, 1/2]^d \to \mathbb{R}$ and approximation accuracy $\varepsilon \in (0, 1/2)$ we construct a ReLU neural network $N_{\varepsilon,f}^{\text{constr}}$ with not more than $c \cdot \varepsilon^{-2(d-1)/\beta}$ non-zero weights and $c' \cdot \log_2(\beta + 2) \cdot (1 + \beta/d)$ layers approximating f up to L^2 error ε. Here c' is an absolute constant, while c might depend on d and β. Furthermore, we show that the scaling behavior of the number of weights with ε is optimal in the previously described sense, i.e., it cannot be improved if one insists that each weight can be encoded using only $\mathcal{O}(\log_2(1/\varepsilon))$ bits. Finally, if $(N_\varepsilon)_{\varepsilon>0}$ is a family of networks (which are *not* required to have encodable weights) meeting this rate for a nonlinear function f, then N_ε needs to have at least $\max\{1, \beta/(3d)\}$ layers, for ε small enough.

The optimality of the approximation rates is derived by lower-bounding the description complexity of the class of piecewise constant functions and by establishing a transference result that yields lower bounds on the sizes of approximating networks.

We observe that the depth of the optimally approximating networks does *not* depend on the approximation accuracy, but is influenced only by the dimension of the input space and by the regularity of the functions. Note that the depth of the networks $N_{\varepsilon,f}^{\text{constr}}$ constructed above coincides (up to a log factor) with the lower bound $\max\{1, \beta/(3d)\}$ from above. This observation offers some explanation for the previously observed efficiency of deep networks: With increasing structure or regularity of the underlying signal class, the best achievable approximation rate gets better, but more depth is required to achieve this optimal approximation rate.

The approximation results can be found in Section 3, and the lower bounds for the number of weights and the number of layers are presented in Section 4. In Section 2, we precisely define the notion of neural networks, and we introduce a kind of calculus for these networks, which in particular covers their composition. This calculus will greatly simplify subsequent proofs.

To not disrupt the flow of the presentation, all results and their interpretations are presented on the first twelve pages of the paper, and almost all proofs have been deferred to the appendix: Appendix A contains the proofs related to Section 3, while the proofs for Section 4 are presented in Appendices B and C. Appendices D and E contain two technical results that are of minor interest, but nevertheless needed for some of our arguments.

Finally, we remark that our construction of approximating neural networks relies on two technical ingredients which are possibly of independent interest for future work:

First, we show (see Lemma A.2) that neural networks can realize an *approximate multiplication*: One can achieve $|xy - N(x, y)| \leq \varepsilon$ using a ReLU neural network N with L layers and $\mathcal{O}(\varepsilon^{-c/L})$ non-zero weights, for a universal constant $c > 0$. A similar result (see Lemma A.4) then holds for general polynomials. We emphasize that it is not a new result that ReLU neural networks can realize an approximate multiplication; this was already observed by Yarotsky [48]. What is new, however, is that *the depth of the network is independent of the approximation accuracy* ε; the depth only influences the approximation rate.

Second, we show (see Lemma A.5) that neural networks can implement a "cutoff", i.e., a multiplication with an indicator function $\chi_{[a_1,b_1]\times\cdots\times[a_d,b_d]}$ using a *fixed number of layers and weights*, as long as the error is measured in L^p, $p < \infty$.

By combining the two results, one sees that neural networks can well approximate every function which is locally well approximated by polynomials.

1.5 Notation

Given a subset $A \subset X$ of a "master set" X (which is usually implied by the context), we define the *indicator function* of A as

$$\chi_A : X \to \{0,1\}, x \mapsto \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \notin A. \end{cases}$$

Moreover, if X is a topological space, we write ∂A for the boundary of A. We denote by $\mathbb{N} = \{1, 2, \dots\}$ the set of natural numbers, by $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ the set of natural numbers including 0, and for $k \in \mathbb{N}$ we denote by $\mathbb{N}_{\geq k}$ all natural numbers larger or equal to k. Occasionally, we also use the notation $\underline{n} := \{1, \dots, n\}$ for $n \in \mathbb{N}$. Furthermore, we write $\lfloor x \rfloor = \max\{k \in \mathbb{Z} : k \leq x\}$ and $\lceil x \rceil = \min\{k \in \mathbb{Z} : k \geq x\}$ for $x \in \mathbb{R}$.

For a function $f : X \to \mathbb{R}$, we write

$$\|f\|_{\sup} := \|f\|_{L^\infty} := \sup_{x \in X} |f(x)| \in [0, \infty].$$

For a given norm $\|\cdot\|$ on $\mathbb{R}^d$, we denote by

$$B_r^{\|\cdot\|}(x) = B_r(x) = \{y \in \mathbb{R}^d : \|y - x\| < r\} \quad \text{and} \quad \overline{B_r}^{\|\cdot\|}(x) = \overline{B_r}(x) = \{y \in \mathbb{R}^d : \|y - x\| \leq r\}$$

the open and closed balls around $x \in \mathbb{R}^d$ of radius $r > 0$. Similar notations are also used in general normed vector spaces, not only in $\mathbb{R}^d$.

For a *multiindex* $\alpha \in \mathbb{N}_0^d$, we write $|\alpha| := \alpha_1 + \cdots + \alpha_d$. This creates a slight ambiguity with the notation $|x|$ for the euclidean norm of $x \in \mathbb{R}^d$, but the context will always make clear which interpretation is desired.

If X, Y, Z are sets and $f : X \to Y$ and $g : Y \to Z$, then we denote by $g \circ f$ the composition of f and g, i.e., $g \circ f(x) = g(f(x))$ for $x \in X$.

We denote by $|M|$ the cardinality $|M| \in \mathbb{N}_0 \cup \{\infty\}$ of a set M. For $A \in \mathbb{R}^{n \times m}$, we denote by $\|A\|_{\ell^0} := |\{(i,j) : A_{i,j} \neq 0\}|$ the number of non-zero entries of A. A similar notation is used for vectors $b \in \mathbb{R}^n$.

2 Neural networks

Below we present a definition of a neural network. For our arguments, it will be crucial to emphasize the difference between a network and the associated function. Thus, we define a network as a structured set of weights and its *realization* as the associated function that results from alternatingly applying the weights and a fixed activation function, which acts componentwise.

Definition 2.1. *Let $d, L \in \mathbb{N}$. A* neural network Φ with input dimension d and L layers *is a sequence of matrix-vector tuples*

$$\Phi = ((A_1, b_1), (A_2, b_2), \dots, (A_L, b_L)),$$

where $N_0 = d$ and $N_1, \dots, N_L \in \mathbb{N}$, and where each A_ℓ is an $N_\ell \times N_{\ell-1}$ matrix, and $b_\ell \in \mathbb{R}^{N_\ell}$.

If Φ is a neural network as above, and if $\varrho : \mathbb{R} \to \mathbb{R}$ is arbitrary, then we define the associated realization of Φ with activation function ϱ *as the map $\mathrm{R}_\varrho(\Phi) : \mathbb{R}^d \to \mathbb{R}^{N_L}$ such that*

$$\mathrm{R}_\varrho(\Phi)(x) = x_L,$$

where x_L results from the following scheme:

$$\begin{aligned} x_0 &:= x, \\ x_\ell &:= \varrho(A_\ell \, x_{\ell-1} + b_\ell), \qquad \text{for } \ell = 1, \dots, L-1, \\ x_L &:= A_L \, x_{L-1} + b_L, \end{aligned}$$

where ϱ acts componentwise, i.e., $\varrho(y) = (\varrho(y^1), \dots, \varrho(y^m))$ for $y = (y^1, \dots, y^m) \in \mathbb{R}^m$.

We call $N(\Phi) := d + \sum_{j=1}^L N_j$ the number of neurons of the network Φ, *$L = L(\Phi)$ the* number of layers, *and finally $M(\Phi) := \sum_{j=1}^L (\|A_j\|_{\ell^0} + \|b_j\|_{\ell^0})$ denotes the number of non-zero entries of all A_ℓ, b_ℓ which we call* the number of weights of Φ. *Moreover, we refer to N_L as the* dimension of the output layer *of Φ.*

To construct new neural networks from existing ones, we will frequently need to concatenate networks or put them in parallel. We first define the concatenation of networks.

Definition 2.2. *Let $L_1, L_2 \in \mathbb{N}$ and let $\Phi^1 = ((A_1^1, b_1^1), \ldots, (A_{L_1}^1, b_{L_1}^1)), \Phi^2 = ((A_1^2, b_1^2), \ldots, (A_{L_2}^2, b_{L_2}^2))$ be two neural networks such that the input layer of Φ^1 has the same dimension as the output layer of Φ^2. Then, $\Phi^1 \bullet \Phi^2$ denotes the following $L_1 + L_2 - 1$ layer network:*

$$\Phi^1 \bullet \Phi^2 := ((A_1^2, b_1^2), \ldots, (A_{L_2-1}^2, b_{L_2-1}^2), (A_1^1 A_{L_2}^2, A_1^1 b_{L_2}^2 + b_1^1), (A_2^1, b_2^1), \ldots, (A_{L_1}^1, b_{L_1}^1)).$$

We call $\Phi^1 \bullet \Phi^2$ the concatenation of Φ^1 and Φ^2.

One directly verifies that $\mathrm{R}_\varrho(\Phi^1 \bullet \Phi^2) = \mathrm{R}_\varrho(\Phi^1) \circ \mathrm{R}_\varrho(\Phi^2)$, which shows that the definition of concatenation is reasonable.

If the activation function $\varrho : \mathbb{R} \to \mathbb{R}$ is the ReLU, i.e., $\varrho(x) = \max\{0, x\}$, then one can construct a simple two-layer network whose realization is the identity on $\mathbb{R}^d$.

Lemma 2.3. *Let ϱ be the ReLU, let $d \in \mathbb{N}$, and define*

$$\Phi^{\mathrm{Id}} := ((A_1, b_1), (A_2, b_2))$$

with

$$A_1 := \begin{pmatrix} \mathrm{Id}_{\mathbb{R}^d} \\ -\mathrm{Id}_{\mathbb{R}^d} \end{pmatrix}, \quad b_1 := 0, \quad A_2 := \begin{pmatrix} \mathrm{Id}_{\mathbb{R}^d} & -\mathrm{Id}_{\mathbb{R}^d} \end{pmatrix}, \quad b_2 := 0.$$

Then $\mathrm{R}_\varrho(\Phi^{\mathrm{Id}}) = \mathrm{Id}_{\mathbb{R}^d}$.

Remark 2.4. *In generalization of Lemma 2.3, for each $d \in \mathbb{N}$, and each $L \in \mathbb{N}_{\geq 2}$, one can construct a network $\Phi_{d,L}^{\mathrm{Id}}$ with L layers and with $2d \cdot L$ nonzero, $\{1, -1\}$-valued weights such that $R_\varrho(\Phi_{d,L}^{\mathrm{Id}}) = \mathrm{Id}_{\mathbb{R}^d}$. In fact, one can choose*

$$\Phi_{d,L}^{\mathrm{Id}} := \left(\left(\begin{pmatrix} \mathrm{Id}_{\mathbb{R}^d} \\ -\mathrm{Id}_{\mathbb{R}^d} \end{pmatrix}, 0 \right), \underbrace{(\mathrm{Id}_{\mathbb{R}^{2d}}, 0), \ldots, (\mathrm{Id}_{\mathbb{R}^{2d}}, 0)}_{L-2 \text{ times}}, \left(\left[\mathrm{Id}_{\mathbb{R}^d} \,|\, -\mathrm{Id}_{\mathbb{R}^d} \right], 0 \right) \right).$$

For $L = 1$, one can achieve the same bounds, simply by setting $\Phi_{d,1}^{\mathrm{Id}} := ((\mathrm{Id}_{\mathbb{R}^d}, 0))$.

Lemma 2.3 enables us to define an alternative concatenation where one can precisely control the number of weights of the resulting network. Note though, that this only works for the ReLU rectifier.

Definition 2.5. *Let $\varrho : \mathbb{R} \to \mathbb{R}$ be the ReLU, let $L_1, L_2 \in \mathbb{N}$, and let $\Phi^1 = ((A_1^1, b_1^1), \ldots, (A_{L_1}^1, b_{L_1}^1))$ and $\Phi^2 = ((A_1^2, b_1^2), \ldots, (A_{L_2}^2, b_{L_2}^2))$ be two neural networks such that the input layer of Φ^1 has the same dimension d as the output layer of Φ^2. Let Φ^{Id} be as in Lemma 2.3.*

Then, the sparse concatenation of Φ^1 and Φ^2 *is defined as*

$$\Phi^1 \odot \Phi^2 := \Phi^1 \bullet \Phi^{\mathrm{Id}} \bullet \Phi^2.$$

Remark 2.6. *It is easy to see that*

$$\Phi^1 \odot \Phi^2 = \left((A_1^2, b_1^2), \ldots, (A_{L_2-1}^2, b_{L_2-1}^2), \left(\begin{pmatrix} A_{L_2}^2 \\ -A_{L_2}^2 \end{pmatrix}, \begin{pmatrix} b_{L_2}^2 \\ -b_{L_2}^2 \end{pmatrix} \right), \left(\left[A_1^1 \,|\, -A_1^1 \right], b_1^1 \right), (A_2^1, b_2^1), \ldots, (A_{L_1}^1, b_{L_1}^1) \right)$$

has $L_1 + L_2$ layers and that $\mathrm{R}_\varrho(\Phi^1 \odot \Phi^2) = \mathrm{R}_\varrho(\Phi^1) \circ \mathrm{R}_\varrho(\Phi^2)$ and $M(\Phi^1 \odot \Phi^2) \leq 2M(\Phi^1) + 2M(\Phi^2)$.

Using concatenations of Φ^{Id}, arbitrarily deep neural networks whose realization is the identity can be constructed. Finally, one can put two networks in parallel by using the following procedure.

Definition 2.7. *Let $L \in \mathbb{N}$ and let $\Phi^1 = ((A_1^1, b_1^1), \ldots, (A_L^1, b_L^1)), \Phi^2 = ((A_1^2, b_1^2), \ldots, (A_L^2, b_L^2))$ be two neural networks with L layers and with d-dimensional input. We define*

$$\mathrm{P}(\Phi^1, \Phi^2) := ((\tilde{A}_1, \tilde{b}_1), \ldots, (\tilde{A}_L, \tilde{b}_L)),$$

where

$$\tilde{A}_1 := \begin{pmatrix} A_1^1 \\ A_1^2 \end{pmatrix}, \quad \tilde{b}_1 := \begin{pmatrix} b_1^1 \\ b_1^2 \end{pmatrix}, \quad \text{and} \quad \tilde{A}_\ell := \begin{pmatrix} A_\ell^1 & 0 \\ 0 & A_\ell^2 \end{pmatrix}, \quad \tilde{b}_\ell := \begin{pmatrix} b_\ell^1 \\ b_\ell^2 \end{pmatrix} \quad \text{for } 1 < \ell \leq L.$$

Then $\mathrm{P}(\Phi^1, \Phi^2)$ is a neural network with d-dimensional input and L layers, called the parallelization of Φ^1 and Φ^2.

One readily verifies that $M(P(\Phi^1,\Phi^2)) = M(\Phi^1) + M(\Phi^2)$, and

$$\mathrm{R}_\varrho(\mathrm{P}(\Phi^1,\Phi^2))(x) = (\mathrm{R}_\varrho(\Phi^1)(x), \mathrm{R}_\varrho(\Phi^2)(x)) \qquad \text{for all } x \in \mathbb{R}^d. \tag{2.1}$$

Remark 2.8. *With the above definition, parallelization is only defined for networks with the same number of layers. However, since we will be working with ReLU networks only, Remark 2.4 and Definition 2.5 enable a reasonable definition of the parallelization of two networks Φ^1, Φ^2 of* different *sizes $L_1 < L_2$: One first sparsely concatenates Φ^1 with a network with $L_2 - L_1$ layers whose realization is the identity, i.e., one defines $\tilde{\Phi}^1 := \Phi^1 \odot \Phi^{\mathrm{Id}}_{d,L_2-L_1}$. We then define $\mathrm{P}(\Phi^1,\Phi^2) := \mathrm{P}(\tilde{\Phi}^1,\Phi^2)$. It is not hard to verify that with this new definition, Equation* (2.1) *still holds. Of course, a similar construction works for $L_1 > L_2$.*

In the sequel, we will be especially interested in neural networks whose weights are quantised since these networks can be stored on a computer. This notion of quantised weights is made precise in the following definition:

Definition 2.9. *Let $\varepsilon \in (0,\infty)$ and let $s \in \mathbb{N}$. A neural network $\Phi = ((A_1,b_1),\dots,(A_L,b_L))$ is said to possess (s,ε)-*quantised weights*, if all weights (i.e., all entries of $A_1,\dots,A_L$ and $b_1,\dots,b_L$) are elements of $[-\varepsilon^{-s},\varepsilon^{-s}] \cap 2^{-s\lceil \log_2(1/\varepsilon)\rceil}\mathbb{Z}$.*

Remark 2.10. *Assume that $\varepsilon \in (0,1/2)$, $p \in (0,\infty)$, $C \geq 1$, $s \in \mathbb{N}$. If Φ is a network with $(s,\varepsilon^p/C)$-quantised weights, then the weights are also $(\tilde{s},\varepsilon)$-quantised, where $\tilde{s} = \lceil ps + s\log_2(C)\rceil + s$. This is because*

$$\varepsilon^{-\tilde{s}} \geq \varepsilon^{-ps-s\log_2(C)} = \varepsilon^{-ps}\left(\frac{1}{\varepsilon}\right)^{s\log_2(C)} \geq \varepsilon^{-ps}2^{s\log_2(C)} = \varepsilon^{-ps}C^s = \left(\frac{\varepsilon^p}{C}\right)^{-s},$$

and

$$\frac{s\cdot\lceil\log_2(1/(\varepsilon^p/C))\rceil}{\tilde{s}\cdot\lceil\log_2(\frac{1}{\varepsilon})\rceil} \leq \frac{s(p\log_2(\frac{1}{\varepsilon}) + \log_2(C) + 1)}{(ps + s\log_2(C) + s)\log_2(\frac{1}{\varepsilon})} = \frac{sp\log_2(\frac{1}{\varepsilon}) + s\log_2(C) + s}{sp\log_2(\frac{1}{\varepsilon}) + s\log_2(C)\log_2(\frac{1}{\varepsilon}) + s\log_2(\frac{1}{\varepsilon})} \leq 1.$$

3 Approximation of classifier functions

In this section, we will provide the main approximation results of the paper. We will only state the results without the underlying proofs, which would otherwise distract from the essentials. All proofs can be found in Appendix A. In this entire section, we assume that $\varrho : \mathbb{R} \to \mathbb{R}, x \mapsto \max\{0,x\}$ is the ReLU.

3.1 Approximation of horizon functions

For $\beta \in (0,\infty)$ with $\beta = n + \sigma$, where $n \in \mathbb{N}_0$ and $\sigma \in (0,1]$ and $d \in \mathbb{N}$, we define for $f \in C^n([-1/2,1/2]^d)$ the norm

$$\|f\|_{C^{0,\beta}} := \max\left\{\max_{|\alpha|\leq n}\|\partial^\alpha f\|_{\sup},\ \max_{|\alpha|=n}\sup_{x,y\in[-1/2,1/2]^d, x\neq y}\frac{|\partial^\alpha f(x) - \partial^\alpha f(y)|}{|x-y|^\sigma}\right\} \in [0,\infty],$$

and for $B > 0$, we define the following class of smooth functions:

$$\mathcal{F}_{\beta,d,B} := \left\{f \in C^n\left([-1/2,1/2]^d\right) : \|f\|_{C^{0,\beta}} \leq B\right\}. \tag{3.1}$$

It should be observed that for $\beta = n+1$, we do *not* require $f \in \mathcal{F}_{\beta,d,B}$ to be $n+1$ times continuously differentiable. Instead, we only require $f \in C^n$, where all derivatives of order n are assumed to be Lipschitz continuous. Of course, if $f \in C^{n+1}([-1/2,1/2]^d)$ with $\|\partial^\alpha f\|_{L^\infty} \leq B$ for all $|\alpha| \leq n+1$, then it easily follows that $\partial^\alpha f$ is Lipschitz continuous, with Lipschitz constant $\mathrm{Lip}(\partial^\alpha f) \leq \sqrt{d}\cdot B$ for all $|\alpha| = n$, so that $f \in \mathcal{F}_{n+1,d,\sqrt{d}B}$. In this sense, our assumptions in case of $\beta = n+1$ are slightly weaker than assuming $f \in C^{n+1}$.

The following theorem establishes optimal approximation rates by ReLU neural networks for the function class $\mathcal{F}_{\beta,d,B}$. It is proved in the appendix as Theorem A.8.

Theorem 3.1. *For any $d \in \mathbb{N}$, and $\beta, B > 0$, there exist constants $c = c(d,\beta,B) > 0$, $s = s(d,\beta,B) \in \mathbb{N}$, and $c' > 0$, such that for any function $f \in \mathcal{F}_{\beta,d,B}$ and any $\varepsilon \in (0,1/2)$, there is a neural network Φ^f_ε with at most $c'\cdot\log_2(2+\beta)\cdot(1+\beta/d)$ layers, and at most $c\cdot\varepsilon^{-d/\beta}$ non-zero, (s,ε)-quantised weights such that for all $p \in (0,2]$,*

$$\|\mathrm{R}_\varrho(\Phi^f_\varepsilon) - f\|_{L^p} < \varepsilon.$$

Remark 3.2. *Approximation of functions in $\mathcal{F}_{\beta,d,B}$ by ReLU networks was already considered in [48, Theorem 1], which provides a result very similar to Theorem 3.1. It differs mainly in two points: First of all, the approximation is with respect to the L^∞ norm in [48, Theorem 1], whereas we provide an approximation result in L^p, $p \in (0,2]$. Additionally, [48, Theorem 1] requires the number of* layers *of the network to grow logarithmically in $1/\varepsilon$, which is not necessary for our result. Overcoming the dependence of the number of layers on ε is achieved by using a refined construction of a multiplication operator, which is given in Lemma A.2, and the fact that (approximate) multiplications with indicator functions can be much more efficiently implemented if only L^2 approximation is required, see Lemma A.5.*

One of the main function classes of interest in the subsequent analysis is that of *horizon functions*. These are $\{0,1\}$-valued functions with a jump along a hypersurface and such that the jump surface is the graph of a smooth function. Formally, we define the class of horizon functions as follows:

Definition 3.3. *Let $d \in \mathbb{N}_{\geq 2}$, and $\beta, B > 0$. Furthermore, let $H := \chi_{[0,\infty)\times\mathbb{R}^{d-1}}$ be the Heaviside function. We define*

$$\mathcal{HF}_{\beta,d,B} := \left\{ f \circ T \in L^\infty\left([-1/2, 1/2]^d\right) : f(x) = H(x_1 + \gamma(x_2,\ldots,x_d), x_2, \ldots, x_d), \gamma \in \mathcal{F}_{\beta,d-1,B}, T \in \Pi(d,\mathbb{R}) \right\},$$

where $\Pi(d,\mathbb{R}) \subset GL(d,\mathbb{R})$ denotes the group of permutation matrices.

Concerning approximation by neural networks of functions in the class $\mathcal{HF}_{\beta,d,B}$, we achieve the following result, which is proved in the appendix as Lemma A.9.

Lemma 3.4. *For any $\beta > 0$, $d \in \mathbb{N}_{\geq 2}$, and $B > 0$ there exists an absolute constant $c' > 0$, and constants $c = c(d,\beta,B) > 0$, and $s = s(d,\beta,B) \in \mathbb{N}$, such that for every function $f \in \mathcal{HF}_{\beta,d,B}$ and every $\varepsilon \in (0, 1/2)$ there is a neural network Φ_ε^f with at most $c' \cdot \log_2(2+\beta) \cdot (1 + \beta/d)$ layers, and at most $c \cdot \varepsilon^{-2(d-1)/\beta}$ non-zero, (s,ε)-quantised weights, such that $\|\mathrm{R}_\varrho(\Phi_\varepsilon^f) - f\|_{L^2([-1/2,1/2]^d)} < \varepsilon$. Moreover, $0 \leq \mathrm{R}_\varrho(\Phi_\varepsilon^f)(x) \leq 1$ for all $x \in [-1/2, 1/2]^d$.*

At first, the approximation of horizon functions might seem a bit arbitrary as this is not a function class of interest that is typically considered. However, this result directly enables the optimal approximation of piecewise constant and even of piecewise smooth functions, as we will see in the next subsection.

3.2 Approximation of piecewise smooth functions

In this subsection, we present approximation rates for piecewise smooth functions f, depending on the smoothness of the jump surfaces and on the smoothness of f on each of the "smooth pieces". We first observe that if one is able to approximate indicator functions χ_K of compact sets $K \subset [-1/2, 1/2]^d$ with say $\partial K \in C^\beta$, then—up to a constant depending on the number N of "pieces"—one can achieve the same approximation quality for functions $f = \sum_{k\leq N} c_k \, \chi_{K_k}$, where $\partial K_k \in C^\beta$ for all $k \leq N$.

Thus, we will only demonstrate how to approximate indicator functions with a condition on the smoothness of the jump surface. We start by introducing a set of domains with smooth boundaries. Let $r \in \mathbb{N}$, $d \in \mathbb{N}_{\geq 2}$, and $\beta, B > 0$. Then we define

$$\mathcal{K}_{r,\beta,d,B} := \left\{ K \subset \left[-\frac{1}{2}, \frac{1}{2}\right]^d : \forall\, x \in \left[-\frac{1}{2}, \frac{1}{2}\right]^d \exists\, f_x \in \mathcal{HF}_{\beta,d,B} : \chi_K = f_x \text{ on } \left[-\frac{1}{2}, \frac{1}{2}\right]^d \cap \overline{B_{2^{-r}}}^{\|\cdot\|_{\ell^\infty}}(x) \right\}.$$

Although the definition of $\mathcal{K}_{r,\beta,d,B}$ is strongly tailored to our needs, it is not overly restrictive. In fact, for every closed set $K' \subset [-1/2, 1/2]^d$ such that $\partial K'$ is locally the graph of a C^β function of all but one coordinate, it follows by compactness of $[-1/2, 1/2]^d$ that $K' \in \mathcal{K}_{r,\beta,d,B}$, for sufficiently large r and large enough B.

We obtain the following approximation result, which is proved in the appendix as Theorem A.10.

Theorem 3.5. *For $r \in \mathbb{N}$, $d \in \mathbb{N}_{\geq 2}$, and $\beta, B > 0$, there are constants $c' > 0$, $c = c(d,r,\beta,B) > 0$, and $s = s(d,r,\beta,B) \in \mathbb{N}$, such that for any $K \in \mathcal{K}_{r,\beta,d,B}$ and any $\varepsilon \in (0, 1/2)$, there is a neural network Φ_ε^K with at most $c' \cdot \log_2(2+\beta) \cdot (1 + \beta/d)$ layers, and at most $c \cdot \varepsilon^{-2(d-1)/\beta}$ non-zero, (s,ε)-quantised weights such that*

$$\|\mathrm{R}_\varrho(\Phi_\varepsilon^K) - \chi_K\|_{L^2} < \varepsilon.$$

Remark 3.6. *Theorem 3.5 establishes approximation rates for piecewise constant functions. It should be noted that the number of required layers is fixed and only depends on the dimension d and the regularity parameter β; in particular, it does* not *depend on the approximation accuracy ε. We will see in Section 4 that the given depth is optimal (up to the factor $c' \cdot \log_2(2+\beta)$) if one wants to achieve the approximation rate stated in the theorem.*

A simple extension of Theorem 3.5 allows us to also approximate piecewise smooth functions optimally. First, let us introduce a suitable class of piecewise smooth functions: For $r \in \mathbb{N}$, $d \in \mathbb{N}_{\geq 2}$, and $\beta, B > 0$ we define $\beta' := (d\beta)/(2(d-1))$ and

$$\mathcal{E}_{r,\beta,d,B} := \{f = \chi_K \cdot g \,:\, g \in \mathcal{F}_{\beta',d,B} \text{ and } K \in \mathcal{K}_{r,\beta,d,B}\}.$$

In terms of this new function class of piecewise smooth functions, we get the following result, which is proven in the appendix as Corollary A.11.

Corollary 3.7. *Let $r \in \mathbb{N}$, $d \in \mathbb{N}_{\geq 2}$, and $B, \beta > 0$. Then there exist constants $c = c(d, \beta, r, B) > 0$, $s = s(d, \beta, r, B) \in \mathbb{N}$, and $c' > 0$, such that for all $\varepsilon \in (0, 1/2)$ and all $f \in \mathcal{E}_{r,\beta,d,B}$ there is a neural network Φ_ε^f with at most $c' \cdot \log_2(2 + \beta) \cdot (1 + \beta/d)$ layers, and at most $c \cdot \varepsilon^{-2(d-1)/\beta}$ non-zero, (s, ε)-quantised weights, such that*

$$\|\mathrm{R}_\varrho(\Phi_\varepsilon^f) - f\|_{L^2} \leq \varepsilon.$$

4 Optimality

In this section, we study two notions of optimality: First of all, we establish in the upcoming subsection a lower bound on the number of weights that neural networks need to have in order to achieve a given approximation accuracy for the class of horizon functions of regularity $\beta > 0$. These results are valid for arbitrary activation functions ϱ. In the second subsection, we study lower bounds on the number of layers that a ReLU neural network needs to have in order to achieve a given approximation rate in terms of the number of weights or neurons. Overall, we will see that the constructions from the previous section achieve the optimal number of weights and have the optimal number of layers, both up to logarithmic factors.

4.1 Optimality in terms of numbers of weights

In this subsection, we show that the approximation results from the preceding section are *sharp*. More precisely, we show that in order to approximate functions from the class $\mathcal{HF}_{\beta,d,B}$ of horizon functions up to an error of $\varepsilon > 0$ (w.r.t. the L^2 norm), one generally needs a network with at least $\Omega(\varepsilon^{-2(d-1)/\beta})$ non-zero weights, independent of the employed activation functions. This claim is still somewhat imprecise; the precise statements are contained in the theorems below. Here, we mention the following four most important points that should be observed:

- We have for all $d \in \mathbb{N}_{\geq 2}$, $r \in \mathbb{N}$, and $\beta, B > 0$ that

  $$\mathcal{HF}_{\beta,d,B} \subset \{\chi_K : K \in \mathcal{K}_{r,\beta,d,B}\} \subset \frac{1}{B} \cdot \mathcal{E}_{r,\beta,d,B}.$$

  Thus all lower bounds established for horizon functions also hold for the function classes of piecewise constant and piecewise smooth functions.

- The statement "one generally needs a network with at least $\Omega(\varepsilon^{-2(d-1)/\beta})$ non-zero weights" suppresses a log factor. Actually, we show that one needs a network with at least $c \cdot \varepsilon^{-2(d-1)/\beta} / \log_2(1/\varepsilon)$ non-zero weights, for a suitable constant $c = c(d, \beta, B) > 0$.

- In [48, Theorem 4], Yarotsky also derives lower bounds for approximating functions using ReLU networks, by using known bounds for the VC dimension of such networks. The most obvious difference of this result to ours is that Yarotsky considers L^∞ approximation of smooth functions, while we consider L^2 approximation of piecewise smooth, possibly discontinuous functions. Apart from these obvious differences, there are also more subtle ones:

  Our lower bounds are more general than those in [48] in the sense that they hold for *arbitrary* activation functions $\varrho : \mathbb{R} \to \mathbb{R}$, as long as $\varrho(0) = 0$, whereas the results of Yarotsky only apply for piecewise linear activation functions with a finite number of "pieces".

  On the other hand, our results are *less* general than those in [48], since we impose (as in [6]) a restriction on the *complexity of the weights* of the network. Put briefly, we assume that each weight of the networks Φ that we consider can be encoded with at most $\lceil C_0 \cdot \log_2(1/\varepsilon)\rceil$ bits, where ε denotes the allowed approximation error, i.e., $\|f - \mathrm{R}_\varrho(\Phi)\|_{L^2} \leq \varepsilon$. This assumption might appear somewhat restrictive and artificial at first glance, but we believe it to be quite natural, for two reasons:

1. The assumption is reasonable if one wants to understand the behavior of networks that are used in practice. Here, the weights of the network have to be stored in the memory of a computer and thus have to be of limited complexity. Note that our results, in particular, apply for the usual floating point numbers, since these only use a fixed number of bits per weight, independent of ε.
2. As noted above, our results apply for general (arbitrary, but fixed) activation functions. In this generality, it is *impossible* to derive nontrivial lower bounds without restricting the size and complexity of the weights: Indeed, [34, Theorem 4] shows that there exists an activation function $\varrho : \mathbb{R} \to \mathbb{R}$ that is analytic, strictly increasing, and sigmoidal (i.e., $\lim_{x\to-\infty} \varrho(x) = 0$ and $\lim_{x\to\infty} \varrho(x) = 1$) such that for any $d \in \mathbb{N}$, any $f \in C([0,1]^d)$ and any $\varepsilon > 0$ there exists a neural network Φ with two hidden layers of dimensions $3d$ and $6d+3$ such that $\|f - \mathrm{R}_\varrho(\Phi)\|_{L^\infty} \leq \varepsilon$. Thus, if one uses this (incredibly complex) activation function ϱ, then one can approximate *arbitrary* continuous functions to an *arbitrary* precision, using a constant number of layers, neurons and weights. From this, it is not too hard to see that a similar result holds for functions in $\mathcal{HF}_{\beta,d,B}$, when the error is measured in L^2. Our bounds show that the weights used in such networks have to be incredibly complex and/or numerically large.

- There are two different settings in which one can derive lower bounds:

  1. For *optimality in a uniform setting*, we are given $\varepsilon > 0$ and want to find the smallest $M_\varepsilon \in \mathbb{N}$ such that for *every* $f \in \mathcal{HF}_{\beta,d,B}$ there is a neural network $\Phi_{\varepsilon,f}$ with at most M_ε non-zero weights (and such that each weight can be encoded with at most $\lceil C_0 \cdot \log_2(1/\varepsilon)\rceil$ bits) satisfying $\|f - \mathrm{R}_\varrho(\Phi_{\varepsilon,f})\|_{L^2} \leq \varepsilon$.

     Put differently, for each sufficiently small $\varepsilon > 0$, there is some "hard to approximate" function $f_\varepsilon \in \mathcal{HF}_{\beta,d,B}$ such that f_ε *cannot* be approximated up to error ε with a network using less than M_ε non-zero weights. In Theorem 4.2, we will show $M_\varepsilon \geq C \cdot \varepsilon^{-2(d-1)/\beta} / \log_2(1/\varepsilon)$ for some $C = C(d, \beta, B, C_0) > 0$.

  2. In the setting of *instance optimality*, we consider for each $f \in \mathcal{HF}_{\beta,d,B}$ the minimal number $M_\varepsilon(f)$ of non-zero weights (of limited complexity, as above) that a neural network needs to have in order to approximate this specific function f up to an L^2 error of at most ε. Note $M_\varepsilon = \sup_{f\in\mathcal{HF}_{\beta,d,B}} M_\varepsilon(f)$.

     Of course, for some f, it can be the case that $M_\varepsilon(f)$ grows much slower than $\varepsilon^{-2(d-1)/\beta}$, for example if the boundary surfaces of f are much smoother than C^β. Indeed, if e.g. $f \in \mathcal{HF}_{\beta+10,d,B}$, then Lemma 3.4 shows $M_\varepsilon(f) \lesssim \varepsilon^{-2(d-1)/(\beta+10)} \lll \varepsilon^{-2(d-1)/\beta}$.

     Now, note that our lower bounds from the preceding point yield for each $\varepsilon \in (0, 1/2)$ a function f_ε with $M_\varepsilon(f_\varepsilon) \geq C \cdot \varepsilon^{-2(d-1)/\beta} / \log_2(1/\varepsilon) \gg \varepsilon^{-\gamma}$, for fixed but arbitrary $\gamma < 2(d-1)/\beta =: \gamma^*$. Nevertheless, since the choice of the function f_ε might depend heavily on the choice of ε, this does *not* rule out the possibility that we could have $M_\varepsilon(f) \in \mathcal{O}(\varepsilon^{-\gamma})$ as $\varepsilon \downarrow 0$ for all $f \in \mathcal{HF}_{\beta,d,B}$ and some $\gamma < \gamma^*$. But as we will see in Theorem 4.3 and in Corollary 4.4, there is a *single function* $f \in \mathcal{HF}_{\beta,d,B}$ such that $M_\varepsilon(f) \notin \mathcal{O}(\varepsilon^{-\gamma})$ for all $\gamma < \gamma^*$.

     This shows that the exponent $\gamma^* = 2(d-1)/\beta$ from Theorem 3.5 is the best possible, not only in a uniform sense, but even for a single (judiciously chosen) function $f \in \mathcal{HF}_{\beta,d,B}$.

After this overview of our optimality results, we state the precise theorems; for the sake of clarity, we deferred the proofs to Appendix B. The first order of business is to make precise the assumption that "the weights of a network can be encoded with K bits".

Definition 4.1. *A* coding scheme for real numbers *is a sequence* $\mathcal{B} = (B_\ell)_{\ell\in\mathbb{N}}$ *of maps* $B_\ell : \{0,1\}^\ell \to \mathbb{R}$.

We say that the coding scheme is consistent *if "each number that can be represented with* ℓ *bits can also be represented with* $\ell + 1$ *bits", i.e., if* $\mathrm{Range}(B_\ell) \subset \mathrm{Range}(B_{\ell+1})$ *for all* $\ell \in \mathbb{N}$.

Given a (not necessarily consistent) coding scheme for real numbers $\mathcal{B} = (B_\ell)_{\ell\in\mathbb{N}}$*, and integers* $M, K \in \mathbb{N}$*, we denote by* $\mathcal{NN}^{\mathcal{B}}_{M,K,d}$ *the class of all neural networks* Φ *with d-dimensional input and one-dimensional output, with at most* M *non-zero weights and such that the value of each nonzero weight of* Φ *is contained in* $\mathrm{Range}(B_K)$*. In words,* $\mathcal{NN}^{\mathcal{B}}_{M,K,d}$ *is the class of all neural networks with at most* M *non-zero weights, each of which can be encoded with* K *bits, using the coding scheme* $\mathcal{B}$*. If the coding scheme is implied by the context, we simply write* $\mathcal{NN}_{M,K,d}$ *instead of* $\mathcal{NN}^{\mathcal{B}}_{M,K,d}$.

Now, given a fixed activation function $\varrho : \mathbb{R} \to \mathbb{R}$ and a fixed coding scheme of real numbers $\mathcal{B}$, it makes sense to ask for a given function $f \in L^2([-1/2, 1/2]^d)$ how quickly the minimal error $\|f - \mathrm{R}_\varrho(\Phi)\|_{L^2}$

(with $\Phi \in \mathcal{NN}_{M,K,d}$) decays, as $M, K \to \infty$. More precisely, given a fixed $C_0 > 0$, we are interested in the behavior of

$$M_\varepsilon(f) := M_\varepsilon^{\mathcal{B},\varrho,C_0}(f) := \inf \left\{ M \in \mathbb{N} \,:\, \exists\, \Phi \in \mathcal{NN}^{\mathcal{B}}_{M,\lceil C_0 \cdot \log_2(1/\varepsilon)\rceil,d} \,:\, \|f - \mathrm{R}_\varrho(\Phi)\|_{L^2} \le \varepsilon \right\}, \qquad (4.1)$$

as $\varepsilon \downarrow 0$. In words, $M_\varepsilon(f)$ describes the minimal number of non-zero weights that a neural network (with activation function ϱ and with weights that can be encoded with $\lceil C_0 \cdot \log_2(1/\varepsilon)\rceil$ bits using the coding scheme $\mathcal{B}$) needs to have in order to approximate f up to an L^2-error of at most ε. Of course, for a badly chosen activation function (e.g., for $\varrho \equiv 0$), it might happen that the set over which the infimum is taken in Equation (4.1) is empty; in this case, $M_\varepsilon(f) := \infty$.

The quantity $M_\varepsilon(f)$ describes how well a *single* function f can be approximated. In contrast, for optimality in a uniform setting, we are given a whole *function class* $\mathcal{C} \subset L^2([-1/2, 1/2]^d)$, and we are interested in the behavior of

$$M_\varepsilon(\mathcal{C}) := M_\varepsilon^{\mathcal{B},\varrho,C_0}(\mathcal{C}) := \sup_{f \in \mathcal{C}} M_\varepsilon^{\mathcal{B},\varrho,C_0}(f)$$

as $\varepsilon \downarrow 0$. Note that $M_\varepsilon(\mathcal{C}) \le M$ if and only if *every* function $f \in \mathcal{C}$ can be approximated with a neural network $\Phi_{f,\varepsilon} \in \mathcal{NN}^{\mathcal{B}}_{M,\lceil C_0 \cdot \log_2(1/\varepsilon)\rceil,d}$ up to an L^2 error of ε.

The following theorem establishes a lower bound on $M_\varepsilon(\mathcal{HF}_{\beta,d,B})$. This lower bound shows that the size of the networks that are constructed in Theorem 3.5 and Corollary 3.7 is optimal, up to a logarithmic factor in $1/\varepsilon$.

Theorem 4.2. *Let $d \in \mathbb{N}_{\ge 2}$ and $\beta, B, C_0 > 0$. Then there exist constants $C = C(d, \beta, B, C_0) > 0$ and $\varepsilon_0 = \varepsilon_0(d, \beta, B) > 0$, such that for each encoding scheme of real numbers $\mathcal{B}$ and any activation function $\varrho : \mathbb{R} \to \mathbb{R}$ with $\varrho(0) = 0$, we have*

$$M_\varepsilon^{\mathcal{B},\varrho,C_0}(\mathcal{HF}_{\beta,d,B}) \ge C \cdot \varepsilon^{-2(d-1)/\beta} \Big/ \log_2(1/\varepsilon) \qquad \text{for all } \varepsilon \in (0, \varepsilon_0).$$

The preceding theorem establishes a lower bound in the uniform setting that was discussed at the beginning of this section. In general, given such a lower bound for the *uniform* error, it is *not* clear that there is also a specific *single* function $f \in \mathcal{HF}_{\beta,d,B}$ for which $M_\varepsilon(f) \gtrsim \varepsilon^{-2(d-1)/\beta}$ (up to log factors). As the following theorem—our main optimality result—shows, this turns out to be true.

Theorem 4.3. *Let $d \in \mathbb{N}_{\ge 2}$, and $\beta, B, C_0 > 0$. Let $\varrho : \mathbb{R} \to \mathbb{R}$ be arbitrary with $\varrho(0) = 0$, and let $\mathcal{B}$ be a* consistent *encoding scheme of real numbers. Then there is some $f \in \mathcal{HF}_{\beta,d,B}$ (potentially depending on $\varrho, d, \mathcal{B}, \beta, B, C_0$) and a null-sequence $(\varepsilon_k)_{k \in \mathbb{N}}$ in $(0, 1/2)$ satisfying*

$$M_{\varepsilon_k}^{\mathcal{B},\varrho,C_0}(f) \ge \frac{\varepsilon_k^{-2(d-1)/\beta}}{\log_2(1/\varepsilon_k) \cdot \log_2(\log_2(1/\varepsilon_k))} \qquad \text{for all } k \in \mathbb{N}.$$

Although it is a trivial consequence of Theorem 4.3, we note the following corollary which shows that the networks constructed in Theorem 3.5 and Corollary 3.7 are of (almost) optimal complexity, even if one is only interested in approximating a single (judiciously chosen) horizon function $f \in \mathcal{HF}_{\beta,d,B}$.

Corollary 4.4. *The function $f \in \mathcal{HF}_{\beta,d,B}$ from Theorem 4.3 satisfies $M_\varepsilon(f) \notin \mathcal{O}(\varepsilon^{-\gamma})$ as $\varepsilon \downarrow 0$, for every $\gamma < 2(d-1)/\beta$.*

Remark. *Thus, the rate obtained in Theorem 3.5 is (almost)* optimal *in the sense that there is one fixed (but unknown) horizon function $f \in \mathcal{HF}_{\beta,d,B}$ such that as $\varepsilon \downarrow 0$, one* cannot *achieve $\|f - \mathrm{R}_\varrho(\Phi_\varepsilon)\|_{L^2} \le \varepsilon$ with a network Φ_ε that has only $\mathcal{O}(\varepsilon^{-\gamma})$ non-zero weights, for some $\gamma < 2(d-1)/\beta$, at least if one insists that the weights of Φ_ε can be encoded with at most $\lceil C_0 \cdot \log_2(1/\varepsilon)\rceil$ bits.*

4.2 We have to go deeper: optimal number of layers

We now establish a lower bound on the number of layers $L(\Phi_\varepsilon)$ that a family of ReLU neural network $(\Phi_\varepsilon)_{\varepsilon > 0}$ needs to have to achieve a given approximation rate for approximating smooth functions. In this whole subsection, we again assume that the activation function ϱ is the ReLU, i.e., $\varrho(x) = \max\{0, x\} = x_+$.

The following theorem is proven in the appendix as Theorem C.6.

Theorem 4.5. *Let $\Omega \subset \mathbb{R}^d$ be nonempty, open, bounded, and connected. Furthermore, let $f \in C^3(\Omega)$ be nonlinear. Then there is a constant $C_f > 0$ satisfying*

$$\|f - R_\varrho(\Phi)\|_{L^p} \geq C_f \cdot (N(\Phi) - 1)^{-L(\Phi)\cdot\left(2+\frac{1}{p}\right)},$$
$$\|f - R_\varrho(\Phi)\|_{L^p} \geq C_f \cdot (M(\Phi) + d)^{-L(\Phi)\cdot\left(2+\frac{1}{p}\right)}$$

for all $1 \leq p < \infty$ and each ReLU neural network Φ with input dimension d and output dimension 1.

Remark 4.6. *The theorem (and also its proof) is inspired by [48, Theorem 6], where it is shown that if $f \in C^2([0,1]^d)$ is nonlinear and $L \in \mathbb{N}$ is fixed, and if $\|f - R_\varrho(\Phi)\|_{L^\infty([0,1]^d)} \leq \varepsilon$ with $\varepsilon \in (0,1)$ for a neural network Φ with $L(\Phi) = L \geq 2$, then $\min\{M(\Phi), N(\Phi)\} \geq c \cdot \varepsilon^{-1/(2(L-1))}$ with $c = c(f, L)$. Note that Yarotsky uses a slightly different definition of neural networks, but the given formulation of his result is already adapted to our definition of neural networks.*

The main difference is that Yarotsky considers approximation in L^∞, while we consider approximation in L^p for $1 \leq p < \infty$, where it is harder to reduce the d-dimensional case to the one-dimensional case, as seen in the proof of Proposition C.5.

Furthermore, there is a difference in the sharpness *of the results: As we saw in Section 3, to approximate a function $f \in \mathcal{F}_{\beta,d,B}$ of regularity C^β up to error ε in the L^p norm, one can take a neural network Φ with $\mathcal{O}\left(\varepsilon^{-d/\beta}\right)$ non-zero weights and a given fixed depth $L \leq c' \log_2(2+\beta)(1 + d/\beta)$ for an absolute constant $c' > 0$. In this sense, up to a logarithmic multiplicative factor, our constructed networks have an* optimal depth.

In contrast, the networks Φ constructed in [48, Theorem 1] to approximate a given function $f \in \mathcal{F}_{n,d,1}$ up to error ε in the L^∞ norm with $\mathcal{O}\left(\varepsilon^{-d/n} \cdot \log_2(1/\varepsilon)\right)$ non-zero weights and neurons have a depth of $\Theta(\log_2(1/\varepsilon))$, i.e., the depth grows with increasing accuracy of the approximation.

Finally, note that it is necessary to assume a certain regularity of f to get the result, since there are nonlinear functions (like the ReLU ϱ) which can be approximated arbitrarily well using ReLU networks with a fixed *number of weights, neurons and layers.*

The following corollary states the connection between the number of weights or neurons and the number of layers more directly. It is proven in the appendix as Corollary C.7.

Corollary 4.7. *Let $\Omega \subset \mathbb{R}^d$ be nonempty, open, bounded, and connected. Furthermore, let $f \in C^3(\Omega)$ be nonlinear. If there are constants $C, \theta > 0$, a null-sequence $(\varepsilon_k)_{k\in\mathbb{N}}$ of positive numbers, and a sequence $(\Phi_k)_{k\in\mathbb{N}}$ of ReLU neural networks satisfying*

$$\|f - R_\varrho(\Phi_k)\|_{L^p} \leq C \cdot \varepsilon_k \quad \text{and} \quad \left[M(\Phi_k) \leq C \cdot \varepsilon_k^{-\theta} \text{ or } N(\Phi_k) \leq C \cdot \varepsilon_k^{-\theta}\right]$$

for all $k \in \mathbb{N}$ and some $1 \leq p < \infty$, then

$$\liminf_{k\to\infty} L(\Phi_k) \geq \frac{p}{2p+1} \cdot \frac{1}{\theta}.$$

Remark 4.8. *The corollary demonstrates that a specific approximation rate in terms of numbers of neurons or weights* cannot *be achieved if the depth of the network is too small. In fact, suppose we are given $f \in \mathcal{E}_{r,\beta,d,B}$ where $r \in \mathbb{N}$, $d \in \mathbb{N}_{\geq 2}$, $\beta, B > 0$ and such that f is non-linear and C^3 when restricted to an open, connected set $A \subset [-1/2, 1/2]^d$, and let $(\varepsilon_k)_{k\in\mathbb{N}}$ be a null-sequence of positive numbers. Then we conclude by Corollary 3.7 that there is a sequence of neural networks Φ_k such that*

$$\|f - R_\varrho(\Phi_k)\|_{L^2} \leq \varepsilon_k \quad \text{and} \quad M(\Phi_k) \leq C \cdot \varepsilon_k^{-\frac{2(d-1)}{\beta}}.$$

for all $k \in \mathbb{N}$. Consequently, Corollary 4.7 applied to f restricted to A demonstrates that there is a lower bound on the number of layers of the constructed networks given—up to a multiplicative factor—by $\beta/(2(d-1))$. We observe that the neural networks constructed in Corollary 3.7 have the optimal numbers of layers, up to a multiplicative factor which is logarithmic in β.

A Approximation of piecewise smooth functions

In this section, we prove all results stated in Section 3, as well as a couple of auxiliary lemmas.

A.1 Approximation of the Heaviside function

As a first step towards approximating horizon functions, it is necessary to recreate a sharp jump. To this end, we show that the Heaviside function can be approximately created with a network of fixed size.

Lemma A.1. *Let $d \in \mathbb{N}_{\geq 2}$ and $H := \chi_{[0,\infty)\times\mathbb{R}^{d-1}}$. For every $\varepsilon > 0$ there exists a neural network Φ^H_ε, with two layers, and five (non-zero) weights which only take values in $\{\varepsilon^{-2}, 1, -1\}$, such that*

$$\|H - \mathrm{R}_\varrho(\Phi^H_\varepsilon)\|_{L^2([-\frac{1}{2},\frac{1}{2}]^d)} \leq \varepsilon.$$

Moreover, $|H(x) - \mathrm{R}_\varrho(\Phi^H_\varepsilon)(x)| \leq \chi_{0\leq x_1 \leq \varepsilon^2}(x)$ for all $x = (x_1, \dots, x_d) \in \mathbb{R}^d$. Finally, $0 \leq \mathrm{R}_\varrho(\Phi^H_\varepsilon) \leq 1$.

Proof. Let $\Phi^H_\varepsilon := ((A_1, b_1), (A_2, b_2))$ with

$$A_1 := \begin{pmatrix} \varepsilon^{-2} & 0 & 0 & \dots \\ \varepsilon^{-2} & 0 & 0 & \dots \end{pmatrix} \in \mathbb{R}^{2\times d}, \quad b_1 := \begin{pmatrix} 0 \\ -1 \end{pmatrix} \in \mathbb{R}^2,$$
$$A_2 := \begin{pmatrix} 1 & -1 \end{pmatrix} \in \mathbb{R}^{1\times 2}, \quad b_2 := 0 \in \mathbb{R}^1.$$

Then

$$\mathrm{R}_\varrho(\Phi^H_\varepsilon)(x) = \varrho\left(\frac{x_1}{\varepsilon^2}\right) - \varrho\left(\frac{x_1}{\varepsilon^2} - 1\right) \text{ for } x = (x_1, \dots, x_d) \in \mathbb{R}^d.$$

From this, we directly compute

$$\mathrm{R}_\varrho(\Phi^H_\varepsilon)(x) = 0 \text{ for } x_1 < 0, \quad \mathrm{R}_\varrho(\Phi^H_\varepsilon)(x) = \frac{x_1}{\varepsilon^2} \text{ for } 0 \leq x_1 \leq \varepsilon^2, \quad \text{and } \mathrm{R}_\varrho(\Phi^H_\varepsilon)(x) = 1 \text{ for } \varepsilon^2 > x_1.$$

We conclude that indeed $|H(x) - R_\varrho(\Phi^H_\varepsilon)(x)| \leq \chi_{0\leq x_1\leq\varepsilon^2}(x)$ and $0 \leq R_\varrho(\Phi^H_\varepsilon) \leq 1$, and therefore

$$\|H - \mathrm{R}_\varrho(\Phi^H_\varepsilon)\|^2_{L^2([-\frac{1}{2},\frac{1}{2}]^d)} \leq \int_{[0,\varepsilon^2]\times[-\frac{1}{2},\frac{1}{2}]^{d-1}} 1\, dx = \varepsilon^2. \qquad \square$$

A.2 Approximation of smooth functions

The second cornerstone of our approximation results is the approximation of smooth functions. The argument proceeds as follows: We start by demonstrating the possibility of approximating a multiplication operator (Lemma A.2) with a ReLU network. With such an operator in place, one can construct networks realizing approximate monomials (Lemma A.3). From there on it is quite clear that for a given function f one is also able to approximate Taylor polynomials at various root points (Lemma A.4). In combination with an approximate partition of unity (Lemmas A.5 and A.6), one can thus approximate C^β functions (Theorem A.8).

We start by constructing the approximate multiplication operator. Already in [48, Proposition 3], it was shown that ReLU networks can compute an approximate multiplication map with error at most ε, using $\log_2(1/\varepsilon)$ layers and nodes. In particular, the number of layers of the network grows indefinitely as $\varepsilon \downarrow 0$. The following lemma offers a compromise between the number of layers and the growth of the number of weights, thereby allowing a construction with a fixed number of layers.

Lemma A.2. *Let $\theta > 0$ be arbitrary. Then, for every $L \in \mathbb{N}$ with $L > 1/\theta$ and each $M \geq 1$, there are constants $c = c(L, M, \theta) \in \mathbb{N}$, $s = s(M) \in \mathbb{N}$, and an absolute constant $c' \in \mathbb{N}$ with the following property:*

For each $\varepsilon \in (0, 1/2)$, there is a neural network $\widetilde{\times}$ with at most $c' \cdot L$ layers, and with at most $c \cdot \varepsilon^{-\theta}$ non-zero, (s, ε)-quantised weights, and such that $\widetilde{\times}$ satisfies the following properties, for all $x, y \in [-M, M]$:

- *We have $|xy - \mathrm{R}_\varrho(\widetilde{\times})(x, y)| \leq \varepsilon$.*
- *We have $\mathrm{R}_\varrho(\widetilde{\times})(x, y) = 0$ if $x \cdot y = 0$.*

Proof. Our proof is heavily based on that of [48, Propositions 2 and 3]. As in that paper, let

$$g : [0, 1] \to [0, 1], \; x \mapsto \begin{cases} 2x, & \text{if } x < \frac{1}{2}, \\ 2(1 - x), & \text{if } x \geq \frac{1}{2}, \end{cases}$$

and for $t \in \mathbb{N}$, let $g_t := \underbrace{g \circ \dots \circ g}_{t \text{ times}}$ be the t-fold composition of g. In the proof of [48, Proposition 2], it was shown that

$$g_t(x) = \begin{cases} 2^t \cdot \left(x - \frac{2k}{2^t}\right), & \text{if } x \in \left[\frac{2k}{2^t}, \frac{2k+1}{2^t}\right], \; k \in \{0, 1, \dots, 2^{t-1} - 1\}, \\ -2^t \cdot \left(x - \frac{2k}{2^t}\right), & \text{if } x \in \left[\frac{2k-1}{2^t}, \frac{2k}{2^t}\right], \; k \in \{1, \dots, 2^{t-1}\}, \end{cases}$$

so that each function g_t is continuous and piecewise affine-linear with 2^t "pieces". From this, it is not hard to see that

$$g_t(x) = 2^t \cdot \left(\varrho(x) + \sum_{k=1}^{2^{t-1}-1} 2 \cdot \varrho\left(x - \frac{2k}{2^t}\right) - \sum_{\ell=1}^{2^{t-1}} 2 \cdot \varrho\left(x - \frac{2\ell-1}{2^t}\right)\right).$$

Therefore, for each $t \in \mathbb{N}$, there is a neural network Φ_t with one-dimensional input and output, with two layers, and $1 + (2^{t-1} + 2^{t-1}) + 1 \le 4 \cdot 2^t$ neurons and at most $2 \cdot 2^{t-1} + 2 \cdot 2^{t-1} + 2^{t-1} + 2^{t-1} \le 4 \cdot 2^t$ non-zero weights, such that $g_t = \mathrm{R}_\varrho(\Phi_t)$. Furthermore, all weights of Φ_t can be chosen to be elements of $[-2^{t+1}, 2^{t+1}] \cap 2^{-t}\mathbb{Z} \subset [-2^{m+1}, 2^{m+1}] \cap 2^{-m}\mathbb{Z}$ for $1 \le t \le m$. Setting $g_0 := \mathrm{id}$, it is easy to see that the same remains true for $t = 0$, cf. Lemma 2.3.

Next, set $s_0 := 1 + \lceil \log_2 M \rceil \in \mathbb{N}$ and define $M_0 := 2^{s_0}$, so that $2M \le M_0 \le 4M$. Furthermore, choose $m := s_0 + \lceil \log_2(1/\varepsilon)/2 \rceil \in \mathbb{N}$, and set $N := \lceil m/L \rceil \in \mathbb{N}$. Now, for each $1 \le t \le m$, we can write $t = kN + r$ for certain $k \in \mathbb{N}_0$ and $r \in \{0, \dots, N-1\}$. Note $k = (t-r)/N \le t/N \le m/N \le L$, and observe $g_t = \underbrace{g_N \circ \cdots \circ g_N}_{k \text{ times}} \circ g_r$, so that we get $g_t = \mathrm{R}_\varrho(\Phi_0^{(t)})$, where $\Phi_0^{(t)} := \underbrace{\Phi_N \odot \cdots \odot \Phi_N}_{k \text{ times}} \odot \Phi_r$ is a neural network with $2(k+1) \le 2(L+1) < 5 \cdot L$ layers. Therefore, with the networks $\Phi^{\mathrm{Id}}_{1,\lambda}$ from Remark 2.4, the network $\Phi^{(t)} := \Phi^{\mathrm{Id}}_{1,5L-2(k+1)} \odot \Phi_0^{(t)}$ satisfies $R_\varrho(\Phi^{(t)}) = g_t$, and $\Phi^{(t)}$ has precisely $5L$ layers, and at most $2^{k+2} \cdot (k \cdot 4 \cdot 2^N + 4 \cdot 2^N + 2 \cdot 5 \cdot L) \le c_1 \cdot 2^N$ nonzero weights, all of which lie in $[-2^{m+1}, 2^{m+1}] \cap 2^{-m}\mathbb{Z}$. Here, $c_1 = c_1(L) > 0$ is a suitable constant, and we used that $k \le L$ and that each network Φ_t with $0 \le t \le N \le m$ has at most $4 \cdot 2^t \le 4 \cdot 2^N$ nonzero weights which all lie in $[-2^{m+1}, 2^{m+1}] \cap 2^{-m}\mathbb{Z}$.

We now use the functions g_t to construct an approximation to the square function. Precisely, in the proof of [48, Proposition 2], it is shown that

$$f_m : [0,1] \to [0,1], x \mapsto x - \sum_{t=1}^m \frac{g_t(x)}{2^{2t}}$$

satisfies $\|(x \mapsto x^2) - f_m\|_{L^\infty} \le 2^{-2-2m}$. Now, set $\Psi := P(\Phi^{\mathrm{Id}}_{1,5L}, P(\Phi^{(1)}, P(\dots, P(\Phi^{(m-1)}, \Phi^{(m)}) \dots)))$, and $\Phi_{\mathrm{sum}} := ((A_{\mathrm{sum}}, 0))$ with $A_{\mathrm{sum}} := (1, -2^{-2}, -2^{-2\cdot 2}, \dots, -2^{-2m}) \in \mathbb{R}^{1 \times (m+1)}$. Then, the neural network $\Phi_0 := \Phi_{\mathrm{sum}} \odot \Psi$ satisfies $R_\varrho(\Phi_0) = f_m$, and Φ_0 has $5L + 1 \le 6L$ layers, and not more than $2(m \cdot c_2 \cdot 2^N + 10L) + 2(m+1) \le c_2 \cdot m \cdot 2^N$ nonzero weights, which all lie in $[-2^{m+1}, 2^{m+1}] \cap 2^{-2m}\mathbb{Z}$. Here, $c_2 = c_2(L) > 0$.

As in the proof of [48, Proposition 3], we now use the polarization identity $x \cdot y = \frac{1}{2}((x+y)^2 - x^2 - y^2)$ and the approximation f_m of the square function to obtain an approximate multiplication. Precisely, define

$$h : [-M_0, M_0]^2 \to \mathbb{R}, (x,y) \mapsto \frac{M_0^2}{2} \cdot \left[f_m\left(\frac{|x+y|}{M_0}\right) - f_m\left(\frac{|x|}{M_0}\right) - f_m\left(\frac{|y|}{M_0}\right)\right].$$

Because of $|x| = \varrho(x) + \varrho(-x)$, and given our implementation of $f_m = R_\varrho(\Phi_0)$, it is easy to see that $h = \mathrm{R}_\varrho(\widetilde{\times})$ for a neural network $\widetilde{\times}$ with at most $10 \cdot L$ layers, and at most $c_3 \cdot m \cdot 2^N$ nonzero weights, all of which are in $[-2^{2m+2s_0}, 2^{2m+2s_0}] \cap 2^{-2m-s_0}\mathbb{Z}$ for some constant $c_3 = c_3(L) \in \mathbb{N}$. Next, since $f_m(0) = 0$, we easily get $\mathrm{R}_\varrho(\widetilde{\times})(x,y) = h(x,y) = 0$ if $x \cdot y = 0$ and $x, y \in [-M, M] \subset [-M_0, M_0]$.

Finally, for $x, y \in [-M, M]$, we have $|x+y| \le |x| + |y| \le 2M \le M_0$, and hence

$$\begin{aligned}
|h(x,y) - xy| &= \left| h(x,y) - M_0^2 \cdot \frac{x}{M_0} \cdot \frac{y}{M_0} \right| \\
(\text{polarization}) &= M_0^2 \left| \frac{1}{2}\left[f_m\left(\frac{|x+y|}{M_0}\right) - f_m\left(\frac{|x|}{M_0}\right) - f_m\left(\frac{|y|}{M_0}\right)\right] - \frac{1}{2}\left[\left(\frac{x}{M_0} + \frac{y}{M_0}\right)^2 - \left(\frac{x}{M_0}\right)^2 - \left(\frac{y}{M_0}\right)^2\right]\right| \\
(\text{since } z^2 = |z|^2) &\le \frac{M_0^2}{2}\left(\left| f_m\left(\frac{|x+y|}{M_0}\right) - \left(\frac{|x+y|}{M_0}\right)^2\right| + \left| f_m\left(\frac{|x|}{M_0}\right) - \left(\frac{|x|}{M_0}\right)^2\right| + \left| f_m\left(\frac{|y|}{M_0}\right) - \left(\frac{|y|}{M_0}\right)^2\right|\right) \\
&\le \frac{M_0^2}{2} \cdot (2^{-2-2m} + 2^{-2-2m} + 2^{-2-2m}) \le \left(\frac{M_0}{2^m}\right)^2 \le \varepsilon.
\end{aligned}$$

Here, the last step used that by choice of m, we have $2^m \ge 2^{s_0} \cdot 2^{\log_2(1/\varepsilon)/2} \ge M_0 \cdot \varepsilon^{-1/2}$. Thus, all that remains to prove is that $\widetilde{\times}$ has the required number of layers and non-zero weights, and that these weights are (s, ε)-quantised for some $s = s(M) \in \mathbb{N}$.

To this end, first recall from above that $\widetilde{\times}$ has at most $10 \cdot L$ layers. Next, we saw above that all weights of $\widetilde{\times}$ lie in $[-2^{2m+2s_0}, 2^{2m+2s_0}] \cap 2^{-2m-s_0}\mathbb{Z}$, where $m = s_0 + \lceil \log_2(1/\varepsilon)/2 \rceil \le 1 + s_0 + 1/2 \log_2(1/\varepsilon)$. Because of $0 < \varepsilon < 1/2$, this implies $2^{2m+2s_0} \le 2^{2+4s_0+\log_2(1/\varepsilon)} = 2^{2+4s_0} \cdot \varepsilon^{-1} \le \varepsilon^{-s}$ for $s := 3 + 4s_0$. Note that indeed $s = s(M) \in \mathbb{N}$, since $s_0 = s_0(M)$. Next, we observe $\log_2(1/\varepsilon) \ge 1$, which implies $2m + s_0 \le 3s_0 + 2 + \log_2(1/\varepsilon) \le (4s_0 + 3) \cdot \log_2(1/\varepsilon) \le s\lceil \log_2(1/\varepsilon) \rceil$, and hence $2^{-2m-s_0}\mathbb{Z} \subset 2^{-s\lceil \log_2(1/\varepsilon)\rceil}\mathbb{Z}$.

Finally, we note that the number $M(\widetilde{\times})$ of non-zero weights of the network $\widetilde{\times}$ satisfies

$$\begin{aligned} M(\widetilde{\times}) &\le c_3 \cdot m \cdot 2^N = c_3 \cdot \left(s_0 + \left\lceil \frac{\log_2(1/\varepsilon)}{2} \right\rceil \right) \cdot 2^{\lceil m/L \rceil} \\ &\le 4c_3 \cdot (1 + s_0) \cdot \log_2(1/\varepsilon) \cdot 2^{m/L} \le 8c_3 \cdot (1 + s_0) 2^{s_0} \cdot \log_2(1/\varepsilon) \cdot 2^{\log_2(1/\varepsilon)/(2L)} \\ &= 8c_3 \cdot (1 + s_0) 2^{s_0} \cdot \log_2(1/\varepsilon) \cdot \varepsilon^{-1/(2L)} \le c_{L,M,\theta} \cdot \varepsilon^{-\theta}. \end{aligned}$$

Here, the last step used that $s_0 = s_0(M)$ and that $1/(2L) < L^{-1} < \theta$, so that $\log_2(1/\varepsilon) \cdot \varepsilon^{-1/(2L)} \le C_{L,\theta} \cdot \varepsilon^{-\theta}$, for a suitable constant $C_{L,\theta} > 0$ and all $\varepsilon \in (0, 1/2)$. ☐

We will be especially interested in the following consequence of Lemma A.2, which demonstrates that monomials can be (approximately) reproduced by neural networks with a fixed number of layers.

Lemma A.3. *Let $n, d, \ell \in \mathbb{N}$ be arbitrary. Then, there is an absolute constant $c' \in \mathbb{N}$ and there are constants $s = s(n) \in \mathbb{N}$, $c = c(d, n, \ell) \in \mathbb{N}$, and $L = L(d, n, \ell) \in \mathbb{N}$ such that $L \le c' \cdot (1 + \lceil \log_2 n \rceil) \cdot (1 + \ell/d)$ with the following property:*

For each $\varepsilon \in (0, 1/2)$ and $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \le n$, there is a neural network Φ_ε^α with d-dimensional input and one-dimensional output, with at most L layers, and with at most $c \cdot \varepsilon^{-d/\ell}$ non-zero, (s, ε)-quantised weights, and such that Φ_ε^α satisfies

$$|\mathrm{R}_\varrho(\Phi_\varepsilon^\alpha)(x) - x^\alpha| \le \varepsilon \quad \text{for all } x \in \left[-\frac{1}{2}, \frac{1}{2}\right]^d. \tag{A.1}$$

Proof. Let $d \in \mathbb{N}$ be fixed, and let $c' \in \mathbb{N}$ and $s = s(2) \in \mathbb{N}$ denote the absolute constants from Lemma A.2 for the choice $M = 2$. We prove the claim by induction over $n \in \mathbb{N}$.

For $n = 1$, we either have $\alpha = 0$, so that $x^\alpha = 1 = R_\varrho(\Phi_\varepsilon^\alpha)(x)$ for a 1-layer network Φ_ε^α that has only one non-zero weight, or there exists $j \in \{1, \dots, d\}$ such that $x^\alpha = x_j$ for all x in $[-1/2, 1/2]^d$. But also in this case, there is a one-layer, one-weight network Φ_ε^α with $\Phi_\varepsilon^\alpha(x) = x_j = x^\alpha$ for all $x \in \mathbb{R}^d$, so that the claim holds.

Now, let us assume that the claim holds for all $1 \le n < k$, for some $k \in \mathbb{N}_{\ge 2}$. We want to show that the claim also holds for $n = k$. First, in case of $|\alpha| < k$, it is easy to see that the claim follows from the one for the case $n = |\alpha| < k$. Therefore, we can assume $|\alpha| = k$. Now, pick $\alpha^{(1)}, \alpha^{(2)} \in \mathbb{N}_0^d$ with $|\alpha^{(2)}| = 2^{\lceil \log_2 k \rceil - 1}$ such that $\alpha^{(1)} + \alpha^{(2)} = \alpha$. Note that indeed $2^{\lceil \log_2 k \rceil - 1} \in \mathbb{N}$ with $2^{\lceil \log_2 k \rceil - 1} < k = |\alpha|$, so that such a choice of $\alpha^{(1)}, \alpha^{(2)}$ is possible. Next, observe $|\alpha^{(1)}| \le |\alpha^{(2)}| < k$, and $\log_2 |\alpha^{(2)}| = \lceil \log_2 k \rceil - 1$.

Thus, by applying the inductive claim with $n = |\alpha^{(2)}|$, we conclude that there are $s_1 = s_1(k) \in \mathbb{N}$, $c_1 = c_1(d, k, \ell) \in \mathbb{N}$, and $L_0 \le c'(1 + \lceil \log_2 k \rceil - 1)(1 + \ell/d)$ such that for all $\varepsilon \in (0, 1/2)$ there exist two neural networks $\Phi_\varepsilon^1, \Phi_\varepsilon^2$ satisfying

$$|\mathrm{R}_\varrho(\Phi_\varepsilon^1)(x) - x^{\alpha^{(1)}}| \le \frac{\varepsilon}{6} \quad \text{and} \quad |\mathrm{R}_\varrho(\Phi_\varepsilon^2)(x) - x^{\alpha^{(2)}}| \le \frac{\varepsilon}{6} \quad \text{for all } x \in \left[-\frac{1}{2}, \frac{1}{2}\right]^d$$

and $\Phi_\varepsilon^1, \Phi_\varepsilon^2$ both have at most L_0 layers, and at most $c_1 \cdot \varepsilon^{-d/\ell}$ non-zero, $(s_1, \varepsilon/6)$-quantised weights. Note by Remark 2.10 that the weights of Φ_ε^1 and Φ_ε^2 are also (s_2, ε)-quantised for a suitable $s_2 = s_2(k) \in \mathbb{N}$. Next, by possibly replacing Φ_ε^t by $\Phi_{1,\lambda_t}^{\mathrm{Id}} \odot \Phi_\varepsilon^t$ with $\Phi_{1,\lambda_t}^{\mathrm{Id}}$ as in Remark 2.4 and for $\lambda_t = L_0 - L(\Phi_\varepsilon^t)$, we can assume that both $\Phi_\varepsilon^1, \Phi_\varepsilon^2$ have exactly L_0 layers. Note in view of Remark 2.6 and because of $L_0 = L_0(d, k, \ell)$ that this will not change the quantisation of the weights, and that the number of weights of Φ_ε^t is still bounded by $c_1' \cdot \varepsilon^{-d/\ell}$ for a suitable $c_1' = c_1'(d, k, \ell)$. For simplicity, we will write c_1 instead of c_1' in what follows.

Now, let $\tilde{\times}$ be the network of Lemma A.2 with accuracy $\delta := \varepsilon/6$ and with $M = 2$, and $\theta = d/\ell$. Note $1/\theta = \ell/d$, so that we can choose $L = 1 + \lfloor \ell/d \rfloor$ in Lemma A.2. Thus, $\tilde{\times}$ can be chosen to have at most $c_2 \cdot \varepsilon^{-d/\ell}$ non-zero, (s, δ)-quantised weights and $c' \cdot (1 + \lfloor \ell/d \rfloor)$ layers, with c', s as chosen at the start of the proof, and for a suitable constant $c_2 = c_2(d, \ell)$. Again by Remark 2.10 we have that the weights of $\tilde{\times}$ are also (s_3, ε)-quantised for a suitable $s_3 = s_3(k) \in \mathbb{N}$.

We now define

$$\Phi_\varepsilon^\alpha := \tilde{\times} \odot P(\Phi_\varepsilon^1, \Phi_\varepsilon^2).$$

By construction, Φ_ε^α has not more than

$$c' \cdot \left(1 + \left\lfloor \frac{\ell}{d} \right\rfloor\right) + L_0 \leq c' + c' \cdot \frac{\ell}{d} + c' \cdot \lceil \log_2 k \rceil \cdot \left(1 + \frac{\ell}{d}\right) \leq c' \cdot (1 + \lceil \log_2 k \rceil) \left(1 + \frac{\ell}{d}\right)$$

many layers, as desired. Next, we estimate by the triangle inequality

$$\begin{aligned}
&|\mathrm{R}_\varrho(\Phi_\varepsilon^\alpha)(x) - x^\alpha| \\
&\quad \leq |\mathrm{R}_\varrho(\tilde{\times})(\mathrm{R}_\varrho(\Phi_\varepsilon^1)(x), \mathrm{R}_\varrho(\Phi_\varepsilon^2)(x)) - \mathrm{R}_\varrho(\Phi_\varepsilon^1)(x)\mathrm{R}_\varrho(\Phi_\varepsilon^2)(x)| + |\mathrm{R}_\varrho(\Phi_\varepsilon^1)(x)\mathrm{R}_\varrho(\Phi_\varepsilon^2)(x) - x^\alpha| \\
&\quad \leq \frac{\varepsilon}{6} + |\mathrm{R}_\varrho(\Phi_\varepsilon^1)(x)\mathrm{R}_\varrho(\Phi_\varepsilon^2)(x) - \mathrm{R}_\varrho(\Phi_\varepsilon^1)(x)x^{\alpha^{(2)}}| + |\mathrm{R}_\varrho(\Phi_\varepsilon^1)(x)x^{\alpha^{(2)}} - x^\alpha| \\
&\quad \leq \frac{\varepsilon}{6} + |\mathrm{R}_\varrho(\Phi_\varepsilon^1)(x)| \cdot \frac{\varepsilon}{6} + |x^{\alpha^{(2)}}| \cdot \frac{\varepsilon}{6} \leq \varepsilon,
\end{aligned}$$

where the last three steps are justified since $x \in [-1/2, 1/2]^d$ and $|\mathrm{R}_\varrho(\Phi_\varepsilon^\ell)(x)| \leq 1 + \varepsilon/6 < 2$ for $\ell \in \{1, 2\}$. Finally, it is easy to see from Remark 2.6 that there exist $c_3 = c_3(d, k, \ell) > 0$ and $s_4 = s_4(k) \in \mathbb{N}$ such that Φ_ε^α has not more than $c_3 \cdot \varepsilon^{-d/\ell}$ non-zero, (s_4, ε)-quantised weights. This concludes the proof. □

A consequence of the ability to reproduce monomials is that we can construct networks that can reproduce polynomials up to a given degree. Moreover, this can be achieved with a fixed and controlled number of layers.

Lemma A.4. *Let $d, m \in \mathbb{N}$, let $B, \beta > 0$, let $\{c_{\ell,\alpha} : \ell = 1, \dots, m, \alpha \in \mathbb{N}_0^d, |\alpha| < \beta\} \subset [-B, B]$ be a sequence of coefficients, and let $(x_\ell)_{\ell=1}^m \subset [-1/2, 1/2]^d$ be a sequence of base points.*

Then, there exist constants $c = c(d, \beta, B) > 0$, $c' \in \mathbb{N}$, $s = s(d, \beta, B) \in \mathbb{N}$, and $L = L(d, \beta) \in \mathbb{N}$ with $L \leq c' \cdot (1 + \lceil \log_2(1 + \beta) \rceil) \cdot (1 + \beta/d)$, such that for all $\varepsilon \in (0, 1/2)$ there is a neural network $\Phi_\varepsilon^{\mathrm{p}}$ with at most $c \cdot (\varepsilon^{-d/\beta} + m)$ many non-zero, (s, ε)-quantised weights, at most L layers, and with an m-dimensional output such that

$$\left|[\mathrm{R}_\varrho(\Phi_\varepsilon^{\mathrm{p}})]_\ell(x) - \sum_{|\alpha| < \beta} c_{\ell,\alpha} \cdot (x - x_\ell)^\alpha\right| < \varepsilon \quad \text{for all } \ell = 1, \dots, m \text{ and } x \in \left[-\frac{1}{2}, \frac{1}{2}\right]^d. \tag{A.2}$$

Proof. Write $\beta = n + \sigma$, with $n \in \mathbb{N}_0$ and $\sigma \in (0, 1]$, let $\{c_{\ell,\alpha} : \ell = 1, \dots, m, \alpha \in \mathbb{N}_0^d, |\alpha| < \beta\} \subset [-B, B]$ and $(x_\ell)_{\ell=1}^m$ be as in the statement of this lemma, and let $\varepsilon \in (0, 1/2)$. By the d-dimensional binomial theorem (cf. [19, Chapter 8, Exercise 2]), we have

$$(x - x_\ell)^\alpha = \sum_{\gamma \leq \alpha} \binom{\alpha}{\gamma} (-x_\ell)^{\alpha - \gamma} x^\gamma \text{ for all } x \in \mathbb{R}^d \text{ and } \alpha \in \mathbb{N}_0^d.$$

Note that $|\alpha| < \beta$ is equivalent to $|\alpha| \leq n$. Thus we have for all $x \in \mathbb{R}^d$ that

$$\sum_{|\alpha| \leq n} c_{\ell,\alpha} (x - x_\ell)^\alpha = \sum_{|\alpha| \leq n} c_{\ell,\alpha} \sum_{\gamma \leq \alpha} \binom{\alpha}{\gamma} x^\gamma (-x_\ell)^{\alpha - \gamma} = \sum_{|\gamma| \leq n} \left[x^\gamma \underbrace{\sum_{\substack{|\alpha| \leq n \\ \alpha \geq \gamma}} c_{\ell,\alpha} \binom{\alpha}{\gamma} (-x_\ell)^{\alpha - \gamma}}_{=: \tilde{c}_{\ell,\gamma}} \right].$$

It is easy to see that there is a constant $C = C(d, \beta, B) \geq 1$ such that for all $\ell \in \{1, \dots, m\}$ and $\gamma \in \mathbb{N}_0^d$ with $|\gamma| \leq n$, we have $|\tilde{c}_{\ell,\gamma}| \leq C$. Furthermore, we just saw that

$$\sum_{|\alpha| \leq n} c_{\ell,\alpha} (x - x_\ell)^\alpha = \sum_{|\gamma| \leq n} \tilde{c}_{\ell,\gamma} \, x^\gamma \quad \text{ for all } x \in \mathbb{R}^d.$$

Since $\varepsilon \in (0, 1/2)$, so that $\varepsilon^{-s} > 2^s$ for $s \in \mathbb{N}$, there clearly exists some $s_1 = s_1(d, \beta, B) \in \mathbb{N}$ (*indep.* of ε) such that there are $\tilde{\tilde{c}}_{\ell,\gamma,\varepsilon} \in [-\varepsilon^{-s_1}, \varepsilon^{-s_1}] \cap 2^{-s_1 \lceil \log_2(1/\varepsilon) \rceil} \mathbb{Z}$ with $|\tilde{c}_{\ell,\gamma} - \tilde{\tilde{c}}_{\ell,\gamma,\varepsilon}| \leq 1$ for all $\gamma \in \mathbb{N}_0^d$ with $|\gamma| \leq n$ and all $1 \leq \ell \leq m$, and such that

$$\left| \sum_{|\gamma| \leq n} \tilde{c}_{\ell,\gamma} \, x^\gamma - \sum_{|\gamma| \leq n} \tilde{\tilde{c}}_{\ell,\gamma,\varepsilon} \, x^\gamma \right| \leq \frac{\varepsilon}{2} \text{ for all } x \in [-1/2, 1/2]^d.$$

Write $\{\gamma \in \mathbb{N}_0^d : |\gamma| \leq n\} = \{\gamma_1, \dots, \gamma_N\}$ with distinct γ_i, for some $N = N(d, n) = N(d, \beta) \in \mathbb{N}$. With this choice, we define for $\ell \in \{1, \dots, m\}$ the network $\Phi^{\ell,\varepsilon} := ((A^{\ell,\varepsilon}, b^\ell))$ where

$$A^{\ell,\varepsilon} := (\tilde{\tilde{c}}_{\ell,\gamma_1,\varepsilon}, \dots, \tilde{\tilde{c}}_{\ell,\gamma_N,\varepsilon}) \in \mathbb{R}^{1 \times N}, \quad \text{ and } \quad b^\ell := 0 \in \mathbb{R}^1.$$

An application of Lemma A.3 (with $\ell = n+1 \in \mathbb{N}$ and with $n+1 \in \mathbb{N}$ instead of n) shows for arbitrary $\delta \in (0, 1/2)$ and $\gamma \in \mathbb{N}_0^d$ with $|\gamma| \leq n+1$ that there exists a network Φ_δ^γ with d-dimensional input and one-dimensional output, at most $c_1 \delta^{-d/(n+1)}$ non-zero, (s_2, δ)-quantised weights and at most L_1 layers, where $c_1 = c_1(d,n) = c_1(d,\beta) > 0$, $s_2 = s_2(n) = s_2(\beta) \in \mathbb{N}$, and $L_1 = L_1(d,n) = L_1(d,\beta) \in \mathbb{N}$ are constants such that $L_1 \leq c' \cdot (1 + \lceil \log_2(n+1) \rceil) \cdot (1 + (n+1)/d)$ for an absolute constant $c' \in \mathbb{N}$, and

$$|\mathrm{R}_\varrho(\Phi_\delta^\gamma)(x) - x^\gamma| \leq \delta \quad \text{ for all } x \in [-1/2, 1/2]^d .$$

We now choose $\delta := \varepsilon/(4CN)$ and define

$$\Phi_\varepsilon^a := P(\Phi^{1,\varepsilon}, P(\Phi^{2,\varepsilon}, \dots, P(\Phi^{m-1,\varepsilon}, \Phi^{m,\varepsilon}))) \quad \text{ and } \quad \Phi_\varepsilon^b := P(\Phi_\delta^{\gamma_1}, P(\Phi_\delta^{\gamma_2}, \dots, P(\Phi_\delta^{\gamma_{N-1}}, \Phi_\delta^{\gamma_N}))).$$

Finally, we set

$$\Phi_\varepsilon^{\mathrm{p}} := \Phi_\varepsilon^a \odot \Phi_\varepsilon^b.$$

By construction, we have that (A.2) holds. Moreover, the weights were chosen quantised (see also Remark 2.10 and note $\delta \geq \varepsilon / C_2$ for a constant $C_2 = C_2(d, \beta, B) > 0$), and the number of weights of Φ_ε^a satisfies $M(\Phi_\varepsilon^a) \leq mN$, while the number of weights of Φ_ε^b—up to a multiplicative constant depending on $n = n(\beta)$, d and B—is bounded by $\varepsilon^{-d/(n+1)} \leq \varepsilon^{-d/\beta}$.

Additionally, since Φ_ε^a has one layer and Φ_ε^b has at most $L_1 \leq c' \cdot (1 + \lceil \log_2(n+1) \rceil) \cdot (1 + (n+1)/d)$ layers, we conclude that $\Phi_\varepsilon^{\mathrm{p}}$ has at most $(1+c') \cdot (1 + \lceil \log_2(n+1) \rceil) \cdot (1 + (n+1)/d)$ layers. Clearly, since $1 \leq n+1 \leq \beta + 1$ there is an absolute constant $c'' \in \mathbb{N}$ such that

$$(1+c') \cdot (1 + \lceil \log_2(n+1) \rceil) \cdot \left(1 + \frac{(n+1)}{d}\right) \leq c'' \cdot (1 + \lceil \log_2(\beta+1) \rceil) \cdot \left(1 + \frac{\beta}{d}\right).$$

This completes the proof. □

The next step of our construction is to demonstrate how to construct a network that (approximately) performs a restriction of an input to an interval.

Lemma A.5. *Let $d \in \mathbb{N}$ and $B \geq 1$, and let $-1/2 \leq a_i \leq b_i \leq 1/2$ for $i = 1, \dots, d$, and let $\varepsilon \in (0, 1/2)$ be arbitrary. Then there exist constants $c = c(d) > 0$, $s = s(d, B) \in \mathbb{N}$, and a neural network Λ_ε with at most four layers, and at most c non-zero, (s, ε)-quantised weights such that for each neural network Φ with one-dimensional output layer and d-dimensional input layer, and with $\|\mathrm{R}_\varrho(\Phi)\|_{L^\infty([-1/2,1/2]^d)} \leq B$, and all $0 < p \leq 2$, we have*

$$\|\mathrm{R}_\varrho(\Lambda_\varepsilon)(\bullet, \mathrm{R}_\varrho(\Phi)(\bullet)) - \chi_{\prod_{i=1}^d [a_i, b_i]} \cdot \mathrm{R}_\varrho(\Phi)\|_{L^p([-\frac{1}{2}, \frac{1}{2}]^d)} \leq \varepsilon.$$

Proof. We set $\tilde{\varepsilon} := 2^{\lfloor \log_2(\varepsilon/(2B\sqrt{2d})) \rfloor} \leq \varepsilon/(2B\sqrt{2d})$. Because of $\varepsilon \in (0, 1/2)$, there exists a constant $s_1 = s_1(d, B) \in \mathbb{N}$ such that there are $\tilde{a}_i, \tilde{b}_i \in [-\varepsilon^{-s_1}, \varepsilon^{-s_1}] \cap 2^{-s_1 \lceil \log_2(1/\varepsilon) \rceil} \mathbb{Z}$ with $|\tilde{a}_i - a_i| < \tilde{\varepsilon}^2$ and $|\tilde{b}_i - b_i| < \tilde{\varepsilon}^2$ for all $i = 1, \dots, d$.

We first construct for each $i \in \{1, \dots, d\}$ the following map t_i, which is clearly the realization of a two layer neural network with at most 12 non-zero, (s_2, ε)-quantised weights, for some $s_2 = s_2(d, B) \in \mathbb{N}$:

$$t_i(x) := \varrho\left(\frac{x - \tilde{a}_i}{\tilde{\varepsilon}^2}\right) - \varrho\left(\frac{x - \tilde{a}_i - \tilde{\varepsilon}^2}{\tilde{\varepsilon}^2}\right) - \varrho\left(\frac{x - \tilde{b}_i + \tilde{\varepsilon}^2}{\tilde{\varepsilon}^2}\right) + \varrho\left(\frac{x - \tilde{b}_i}{\tilde{\varepsilon}^2}\right) \quad \text{ for } x \in \left[-\frac{1}{2}, \frac{1}{2}\right].$$

A simple computation yields that if $\tilde{b}_i - \tilde{a}_i > 2\tilde{\varepsilon}^2$ then

$$t_i(x) = \begin{cases} 0 & \text{for } x \in \mathbb{R} \setminus [\tilde{a}_i, \tilde{b}_i], \\ \frac{x - \tilde{a}_i}{\tilde{\varepsilon}^2} & \text{for } x \in [\tilde{a}_i, \tilde{a}_i + \tilde{\varepsilon}^2], \\ 1 & \text{for } x \in [\tilde{a}_i + \tilde{\varepsilon}^2, \tilde{b}_i - \tilde{\varepsilon}^2], \\ 1 - \frac{x - (\tilde{b}_i - \tilde{\varepsilon}^2)}{\tilde{\varepsilon}^2} & \text{for } x \in [\tilde{b}_i - \tilde{\varepsilon}^2, \tilde{b}_i]. \end{cases}$$

We continue defining the function $n_\varepsilon : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ which will be the realization of Λ_ε. Let $B_0 := 2^{\lceil \log_2 B \rceil}$. If $\tilde{b}_i - \tilde{a}_i > 2\tilde{\varepsilon}^2$ holds for all $i = 1, \dots, d$ then we define

$$n_\varepsilon(x, y) := B_0 \cdot \varrho\left(\sum_{i=1}^d t_i(x_i) + \varrho\left(\frac{y}{B_0}\right) - d\right) - B_0 \cdot \varrho\left(\sum_{i=1}^d t_i(x_i) + \varrho\left(-\frac{y}{B_0}\right) - d\right).$$

Otherwise (i.e., if $\tilde{b}_i - \tilde{a}_i \leq 2\tilde{\varepsilon}^2$ for some $i \in \{1, \dots, d\}$), we set $n_\varepsilon \equiv 0$. In both cases, it is easy to see that n_ε is the realization of a four layer neural network Λ_ε with at most $c = c(d)$ non-zero, (s_3, ε)-quantised weights, for some $s_3 = s_3(d,B) \in \mathbb{N}$. Further, in both cases, the following holds for all $y \in [-B,B] \subset [-B_0,B_0]$: If $x \in \prod_{i=1}^d [\tilde{a}_i + \tilde{\varepsilon}^2, \tilde{b}_i - \tilde{\varepsilon}^2]$ then $n_\varepsilon(y) = y$, and if $x \in \mathbb{R}^d \setminus \prod_{i=1}^d [\tilde{a}_i, \tilde{b}_i]$, then $n_\varepsilon(y) = 0$. Moreover, $\prod_{i=1}^d [\tilde{a}_i, \tilde{b}_i] \setminus \prod_{i=1}^d [\tilde{a}_i + \tilde{\varepsilon}^2, \tilde{b}_i - \tilde{\varepsilon}^2]$ has Lebesgue measure bounded by $2d\tilde{\varepsilon}^2 \le \varepsilon^2/(2B)^2$. Finally, it is not hard to see (using the monotonicity of $\varrho(x) = x_+$) that $-B \le -\varrho(y) \le n_\varepsilon(x,y) \le \varrho(y) \le B$ for arbitrary $y \in [-B,B]$. Therefore, for any measurable $f : [-1/2, 1/2]^d \to [-B,B]$, we have

$$\begin{aligned}\|n_\varepsilon(\bullet, f(\bullet)) - \chi_{\prod_{i=1}^d [a_i,b_i]} \cdot f\|_{L^2} \le & \|n_\varepsilon(\bullet, f(\bullet)) - \chi_{\prod_{i=1}^d [\tilde{a}_i,\tilde{b}_i]} \cdot f\|_{L^2} \\ & + \|\chi_{\prod_{i=1}^d [\tilde{a}_i,\tilde{b}_i]} \cdot f - \chi_{\prod_{i=1}^d [a_i,b_i]} \cdot f\|_{L^2}.\end{aligned}$$

By the previous considerations, and since $|f| \le B$, we can estimate

$$\|n_\varepsilon(\bullet, f(\bullet)) - \chi_{\prod_{i=1}^d [\tilde{a}_i,\tilde{b}_i]} \cdot f\|_{L^2} \le \left(\frac{\varepsilon^2}{(2B)^2}\right)^{\frac{1}{2}} \cdot B \le \frac{\varepsilon}{2}.$$

Since $2d\tilde{\varepsilon}^2 \le \varepsilon^2/(2B)^2$, and since $|\tilde{a}_i - a_i| \le \tilde{\varepsilon}^2$ and $|\tilde{b}_i - b_i| \le \tilde{\varepsilon}^2$ for all $i \in \{1,\dots,d\}$, we also have

$$\|\chi_{\prod_{i=1}^d [\tilde{a}_i,\tilde{b}_i]} \cdot f - \chi_{\prod_{i=1}^d [a_i,b_i]} \cdot f\|_{L^2} \le \left(\frac{\varepsilon^2}{(2B)^2}\right)^{\frac{1}{2}} \cdot B \le \frac{\varepsilon}{2}.$$

In combination these estimates imply the result (for the case $p = 2$), by choosing $f = R_\varrho(\Phi)$. Since $[-1/2, 1/2]^d$ (with the Lebesgue measure) is a probability space, the claim for $0 < p < 2$ follows from Jensen's inequality. ☐

For technical reasons we require the following refinement of Lemma A.5.

Lemma A.6. *Let $d, m, s \in \mathbb{N}$, and $\varepsilon \in (0, 1/2)$, and let Φ be a neural network with m-dimensional output and d-dimensional input, with (s,ε)-quantised weights, and such that $\|[\mathrm{R}_\varrho(\Phi)]_\ell\|_{L^\infty([-1/2,1/2]^d)} \le B$ for some $B \ge 1$ and all $\ell = 1,\dots,m$. Let $-1/2 \le a_{i,\ell} \le b_{i,\ell} \le 1/2$ for $i = 1,\dots,d$ and $\ell = 1,\dots,m$.*

Then, there exist constants $c = c(d) > 0$, $s_0 = s_0(d,B) \in \mathbb{N}$, and a neural network Ψ_ε with d-dimensional input layer and 1-dimensional output layer, with at most $6 + L(\Phi)$ layers, and at most $c \cdot (m + L(\Phi) + M(\Phi))$ non-zero, $(\max\{s, s_0\}, \varepsilon/m)$-quantised weights, such that

$$\left\| \mathrm{R}_\varrho(\Psi_\varepsilon) - \sum_{\ell=1}^m \chi_{\prod_{i=1}^d [a_{i,\ell}, b_{i,\ell}]} \cdot [\mathrm{R}_\varrho(\Phi)]_\ell \right\|_{L^2([-\frac{1}{2},\frac{1}{2}]^d)} \le \varepsilon.$$

Proof. For each $\ell \in \{1,\dots,m\}$ let Λ_ε^ℓ be the neural network provided by Lemma A.5 applied with $a_i = a_{i,\ell}$, $b_i = b_{i,\ell}$ and ε/m instead of ε. There exists $c_0 = c_0(d)$, $s_0 = s_0(d,B)$ such that Λ_ε^ℓ has at most c_0 nonzero, $(s_0, \varepsilon/m)$-quantised weights and four layers. Let $P_\ell \in \mathbb{R}^{(d+1)\times(d+m)}$ be the matrix associated (via the standard basis) to the linear map $\mathbb{R}^d \times \mathbb{R}^m \ni (x,y) \mapsto (x, y_\ell) \in \mathbb{R}^d \times \mathbb{R}^1$, and let $\Phi_\ell := ((P_\ell, 0))$ be the associated 1-layer network. Clearly, Φ_ℓ has $d+1$ nonzero, $(1,\varepsilon)$-quantised weights.

Next, let $L := L(\Phi)$, and set $\widetilde{\Phi} := P(\Phi^{\mathrm{Id}}_{d,L}, \Phi)$, where $\Phi^{\mathrm{Id}}_{d,L}$ is as in Remark 2.4, so that $\Phi^{\mathrm{Id}}_{d,L}$ has $L = L(\Phi)$ layers and $2d \cdot L(\Phi)$ nonzero, $(1,\varepsilon)$-quantised weights, and $\mathrm{R}_\varrho(\Phi^{\mathrm{Id}}_{d,L}) = \mathrm{Id}_{\mathbb{R}^d}$. We conclude that $\widetilde{\Phi}$ has $L(\Phi)$ layers, and at most $M(\Phi) + 2dL(\Phi)$ nonzero, (s,ε)-quantised weights.

Additionally, we define $\Phi^{\mathrm{sum}} := ((A^{\mathrm{sum}}, b^{\mathrm{sum}}))$ where

$$A^{\mathrm{sum}} := (1,1,\dots,1) \in \mathbb{R}^{1\times m}, \quad \text{and} \quad b^{\mathrm{sum}} := 0.$$

Φ^{sum} has exactly m non-zero, $(1,\varepsilon)$-quantised weights and one layer.

Finally, define

$$\Psi_\varepsilon := \Phi^{\mathrm{sum}} \odot P(\Lambda_\varepsilon^1 \odot \Phi_1, P(\dots, P(\Lambda_\varepsilon^{m-1} \odot \Phi_{m-1}, \Lambda_\varepsilon^m \odot \Phi_m))) \odot \widetilde{\Phi}.$$

By Remark 2.6 we have that Ψ_ε has $1 + 5 + L(\Phi)$ layers and at most

$$4(m + m \cdot 2(d + 1 + c_0) + M(\Phi) + 2dL(\Phi)) \le c \cdot (m + L(\Phi) + M(\Phi))$$

nonzero, $(\max\{s_0, s\}, \varepsilon/m)$ weights for a constant $c = c(d) > 0$.

We observe that

$$R_\varrho(\Psi_\varepsilon)(x) = \sum_{\ell=1}^m R_\varrho(\Lambda_\varepsilon^\ell)\,(x, [R_\varrho(\Phi)(x)]_\ell) \quad \text{for } x \in \mathbb{R}^d$$

and hence by the triangle inequality

$$\begin{aligned}
&\left\| R_\varrho(\Psi_\varepsilon) - \sum_{\ell \le m} \chi_{\prod_{i=1}^d [a_{i,\ell}, b_{i,\ell}]} \cdot [\mathrm{R}_\varrho(\Phi)]_\ell \right\|_{L^2([-\frac{1}{2},\frac{1}{2}]^d)} \\
&\le \sum_{\ell \le m} \left\| R_\varrho(\Lambda_{\ell,\varepsilon})\left(\bullet, [R_\varrho(\Phi)(\bullet)]_\ell\right) - \chi_{\prod_{i=1}^d [a_{i,\ell}, b_{i,\ell}]} \cdot [\mathrm{R}_\varrho(\Phi)]_\ell \right\|_{L^2([-\frac{1}{2},\frac{1}{2}]^d)} \overset{(*)}{\le} \sum_{\ell \le m} \frac{\varepsilon}{m} = \varepsilon,
\end{aligned}$$

where the step marked with $(*)$ holds by choice of the neural networks Λ_ε^ℓ, see Lemma A.5. □

Our next larger goal is to show that neural networks can well approximate smooth functions with respect to the L^2 norm, in such a way that the number of layers does *not* grow with the approximation accuracy, only with the smoothness of the function. A central ingredient for the proof is the local approximation of smooth functions via their Taylor polynomials. Precisely, we need the following result, which is probably folklore:

Lemma A.7. *Let $\beta \in (0, \infty)$, and write $\beta = n + \sigma$ with $n \in \mathbb{N}_0$ and $\sigma \in (0, 1]$, and let $d \in \mathbb{N}$. Then there is a constant $C = C(\beta, d) > 0$ with the following property:*

For each $f \in \mathcal{F}_{\beta,d,B}$ and arbitrary $x_0 \in (-1/2, 1/2)^d$, there is a polynomial $p(x) = \sum_{|\alpha| \le n} c_\alpha (x - x_0)^\alpha$ with $c_\alpha \in [-C \cdot B, C \cdot B]$ for all $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \le n$ such that

$$|f(x) - p(x)| \le C \cdot B \cdot |x - x_0|^\beta \qquad \text{for all } x \in [-1/2, 1/2]^d .$$

In fact, $p = p_{f,x_0}$ is the Taylor polynomial of f of degree n.

Proof. In case of $n = 0$, the claim is trivial: If we set $p(x) := f(x_0)$, then $|f(x_0)| \le \|f\|_{C^{0,\beta}} \le B$, and

$$|f(x) - p(x)| = |f(x) - f(x_0)| \le \|f\|_{C^{0,\beta}} \cdot |x - x_0|^\sigma \le B \cdot |x - x_0|^\beta,$$

as desired. Therefore, we can from now on assume $n \in \mathbb{N}$.

In the following, we use a slightly different multi-index notation, to be compatible with the notation in [33]: We write $\underline{d} := \{1, \dots, d\}$, and for $I = (i_1, \dots, i_m) \in \underline{d}^m$ with $m \in \mathbb{N}$, we write $\partial_I f := \partial_{i_1} \cdots \partial_{i_m} f$ and $y^I = y^{i_1} \cdots y^{i_m}$ for $y \in \mathbb{R}^d$. Using this notation, the Taylor polynomial of f of degree $n-1$ at x_0 is given by

$$p_0(x) := f(x_0) + \sum_{m=1}^{n-1} \frac{1}{m!} \sum_{I \in \underline{d}^m} (\partial_I f)(x_0) \cdot (x - x_0)^I .$$

Taylor's theorem with integral remainder (see [33, Theorem C.15]) shows for $x \in (-1/2, 1/2)^d$ that

$$\begin{aligned}
&f(x) - p_0(x) \\
&= \frac{1}{(n-1)!} \cdot \sum_{I \in \underline{d}^n} (x - x_0)^I \int_0^1 (1-t)^{n-1} \partial_I f(x_0 + t(x - x_0)) dt \\
&= \frac{1}{(n-1)!} \Bigg(\sum_{I \in \underline{d}^n} (x - x_0)^I \int_0^1 (1-t)^{n-1} \partial_I f(x_0) dt \\
&\qquad + \sum_{I \in \underline{d}^n} (x - x_0)^I \int_0^1 (1-t)^{n-1} [\partial_I f(x_0 + t(x - x_0)) - \partial_I f(x_0)] dt \Bigg) \\
&= \frac{1}{n!} \cdot \sum_{I \in \underline{d}^n} \partial_I f(x_0) \cdot (x - x_0)^I + \frac{1}{(n-1)!} \sum_{I \in \underline{d}^n} (x - x_0)^I \int_0^1 (1-t)^{n-1} [\partial_I f(x_0 + t(x - x_0)) - \partial_I f(x_0)] dt \\
&=: q(x) + R(x).
\end{aligned}$$

But $p := p_0 + q$ is the Taylor polynomial of f of degree n at x_0, and $p(x) = \sum_{|\alpha| \le n} c_\alpha (x - x_0)^\alpha$ for certain $c_\alpha \in \mathbb{R}$, which are easily seen to satisfy

$$|c_\alpha| \le \sum_{I \in \underline{d}^{|\alpha|} \text{ with } \alpha = i_1 + \dots + i_{|\alpha|}} |\partial_I f(x_0)| \le d^{|\alpha|} \cdot B \le d^n \cdot B.$$

Finally, since $\partial_I f$ is σ Hölder continuous with $\mathrm{Lip}_\sigma(\partial_I f) \leq \|f\|_{C^{0,\beta}} \leq B$ for $I \in \underline{d}^n$, we get

$$|f(x) - p(x)| = |R(x)| \leq \frac{1}{(n-1)!} \cdot \sum_{I \in \underline{d}^n} |(x - x_0)^I| \cdot \int_0^1 (1-t)^{n-1} B \cdot |t(x - x_0)|^\sigma dt$$
$$\leq \frac{d^n}{n!} |x - x_0|^n \cdot B \cdot |x - x_0|^\sigma \leq C \cdot B \cdot |x - x_0|^\beta$$

for a suitable constant $C = C(d, n) = C(d, \beta)$. By continuity, this estimate holds for all $x \in [-1/2, 1/2]^d$, not just for $x \in (-1/2, 1/2)^d$. □

Now, we can finally prove our main result about the L^2-approximation of smooth functions using ReLU networks.

Theorem A.8. *For any $d \in \mathbb{N}$, and $\beta, B > 0$, there exist constants $c = c(d, \beta, B) > 0$, $s = s(d, \beta, B) \in \mathbb{N}$, $c' > 0$, and $L = L(\beta, d) \in \mathbb{N}$ with $L \leq c' \cdot (1 + \lceil \log_2(1 + \beta) \rceil) \cdot (1 + \beta/d)$, such that for any function $f \in \mathcal{F}_{\beta,d,B}$ and any $\varepsilon \in (0, 1/2)$, there is a neural network Φ_ε^f with at most L layers, and at most $c \cdot \varepsilon^{-d/\beta}$ non-zero, (s, ε)-quantised weights such that for all $p \in (0, 2]$,*

$$\|\mathrm{R}_\varrho(\Phi_\varepsilon^f) - f\|_{L^p([-1/2,1/2]^d)} < \varepsilon.$$

Proof. As in the proof of Lemma A.5, it suffices to consider the case $p = 2$, thanks to Jensen's inequality. Also, we only need to consider the case $B = 1$, since the general case follows by reweighting. Let $\beta = n + \sigma$ with $n \in \mathbb{N}_0$ and $\sigma \in (0, 1]$. Further, let $N \in \mathbb{N}$ be arbitrary, and set for $\lambda \in \{1, \ldots, N\}^d$

$$I_\lambda := \prod_{i=1}^d \left[\frac{\lambda_i - 1}{N} - \frac{1}{2}, \frac{\lambda_i}{N} - \frac{1}{2} \right].$$

As a result, we have (with disjointness up to null-sets) that

$$\dot{\bigcup_{\lambda \in \{1,\ldots,N\}^d}} I_\lambda = \left[-\frac{1}{2}, \frac{1}{2} \right]^d, \text{ and } I_\lambda \subset \overline{B}_{\frac{1}{N}}^{\|\cdot\|_{\ell^\infty}}(x) \subset \overline{B}_{\frac{d}{N}}^{|\cdot|}(x), \text{ for all } x \in I_\lambda. \tag{A.3}$$

Choose for all $\lambda \in \{1, \ldots, N\}^d$ a point x_λ in the interior of I_λ, and set $c_{\lambda,\alpha} := \partial^\alpha f(x_\lambda)/\alpha!$ for $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \leq n$. Note $|c_{\lambda,\alpha}| \leq B$. Then we take $\Phi_{\varepsilon/4}^{\mathrm{p}}$ as in Lemma A.4 with accuracy $\varepsilon/4$ and $m = N^d$, and with $B_0 := \max\{1, 2^{\lceil \log_2 B \rceil}\}$ instead of B. By Lemma A.4, $\Phi_{\varepsilon/4}^{\mathrm{p}}$ has at most $L_1 = L_1(d, \beta)$ layers, with $L_1 \leq c' \cdot (1 + \lceil \log_2(1 + \beta) \rceil) \cdot (1 + \beta/d)$, and at most $c_1(\varepsilon^{-d/\beta} + N^d)$ non-zero, (s, ε)-quantised weights (see also Remark 2.10), for certain $s = s(d, \beta, B) \in \mathbb{N}$, $c_1 = c_1(d, \beta, B) > 0$, and an absolute constant $c' \in \mathbb{N}$. We compute

$$\left\| f - \sum_{\lambda \in \{1,\ldots,N\}^d} \chi_{I_\lambda} [\mathrm{R}_\varrho(\Phi_{\varepsilon/4}^{\mathrm{p}})]_\lambda \right\|_{L^\infty} \leq \sup_{\substack{\lambda \in \{1,\ldots,N\}^d \\ x \in I_\lambda}} |f(x) - [\mathrm{R}_\varrho(\Phi_{\varepsilon/4}^{\mathrm{p}})]_\lambda(x)|.$$

By construction of $\Phi_{\varepsilon/4}^{\mathrm{p}}$, we conclude that

$$\sup_{\substack{\lambda \in \{1,\ldots,N\}^d \\ x \in I_\lambda}} \left| f(x) - [\mathrm{R}_\varrho(\Phi_{\varepsilon/4}^{\mathrm{p}})]_\lambda(x) \right| \leq \sup_{\substack{\lambda \in \{1,\ldots,N\}^d \\ x \in I_\lambda}} \left| f(x) - \sum_{|\alpha| \leq n} \frac{\partial^\alpha f(x_\lambda)}{\alpha!} (x - x_\lambda)^\alpha \right| + \frac{\varepsilon}{4}. \tag{A.4}$$

Now, note for $y \in \mathbb{R}$ that $\max\{y, -B_0\} = \varrho(y + B_0) - B_0$ and $\min\{y, B_0\} = B_0 - \varrho(B_0 - y)$. Therefore, we can slightly modify the neural network $\Phi_{\varepsilon/4}^{\mathrm{p}}$ to obtain a network $\Psi_{\varepsilon/4}^{\mathrm{p}}$ with essentially* the same complexity as $\Phi_{\varepsilon/4}^{\mathrm{p}}$ and such that $[R_\varrho(\Psi_{\varepsilon/4}^{\mathrm{p}})]_\lambda = \min\{B_0, \max\{-B_0, [R_\varrho(\Phi_{\varepsilon/4}^{\mathrm{p}})]_\lambda\}\}$ for all $\lambda \in \{1, \ldots, N\}^d$. Since we have $-B_0 \leq -B \leq f \leq B \leq B_0$, it is not hard to see that (A.4) remains valid also for $\Psi_{\varepsilon/4}^{\mathrm{p}}$. In the following, we will thus simply write $\Phi_{\varepsilon/4}^{\mathrm{p}}$ for $\Psi_{\varepsilon/4}^{\mathrm{p}}$.

Using Lemma A.7 and Equation (A.3) we obtain an absolute constant $C = C(d, \beta) > 0$ such that

$$\sup_{\substack{\lambda \in \{1,\ldots,N\}^d \\ x \in I_\lambda}} \left| f(x) - \sum_{|\alpha| \leq n} \frac{\partial^\alpha f(x_\lambda)}{\alpha!} (x - x_\lambda)^\alpha \right| \leq CB \left(\frac{d}{N} \right)^\beta.$$

*Precisely, the constants c', c_1, s can be slightly enlarged in such a way that $\Psi_{\varepsilon/4}^{\mathrm{p}}$ satisfies the same complexity estimates as $\Phi_{\varepsilon/4}^{\mathrm{p}}$.

Now, choose

$$N := \left\lceil \left(\frac{\varepsilon}{4CBd^{\beta}}\right)^{-\frac{1}{\beta}} \right\rceil, \quad \text{so that} \quad \sup_{\substack{\lambda \in \{1,\dots,N\}^d \\ x \in I_\lambda}} |f(x) - [\mathrm{R}_\varrho(\Phi^{\mathrm{p}}_{\varepsilon/4})]_\lambda(x)| \leq \frac{\varepsilon}{2}.$$

By the triangle inequality, we see that we are done if there is a network Ψ_ε with quantised weights, at most $L \leq c'' \cdot (1 + \lceil \log_2(1+\beta) \rceil) \cdot (1 + \beta/d)$ layers for an absolute constant c'', and at most $c \cdot \varepsilon^{-d/\beta}$ non-zero weights satisfying $\|\mathrm{R}_\varrho(\Psi_\varepsilon) - \sum_{\lambda \in \{1,\dots,N\}^d} \chi_{I_\lambda} [\mathrm{R}_\varrho(\Phi^{\mathrm{p}}_{\varepsilon/4})]_\lambda\|_{L^2} \leq \varepsilon/2$, which we verify by applying Lemma A.6.

Indeed, if we apply that lemma, with $\varepsilon/2$ instead of ε, with $\Phi = \Phi^{\mathrm{p}}_{\varepsilon/4}$ and $m = N^d$, and with the intervals I_λ, $\lambda \in \{1,\dots,N\}^d$, then we get a neural network Ψ_ε which satisfies the desired estimate.

Furthermore, Ψ_ε has $6 + L(\Phi^{\mathrm{p}}_{\varepsilon/4}) \leq 6 + L_1 \leq c'' \cdot (1 + \lceil \log_2(1+\beta) \rceil) \cdot (1 + \beta/d)$ layers, for an absolute constant $c'' > 0$. Moreover, Ψ_ε has at most

$$c \cdot (N^d + L(\Phi^{\mathrm{p}}_{\varepsilon/4}) + M(\Phi^{\mathrm{p}}_{\varepsilon/4})) \leq c \cdot (N^d + L_1 + M(\Phi^{\mathrm{p}}_{\varepsilon/4})) \leq c_2 \cdot (N^d + c_1(\varepsilon^{-d/\beta} + N^d))$$

non-zero, $(\max\{s, s_0\}, \varepsilon/(2N^d))$-quantised weights, with constants $c = c(d) > 0$, $c_2 = c_2(d,\beta,B) > 0$, and $s_0 = s_0(d,B) \in \mathbb{N}$. By choice of N, this shows that Ψ_ε has the correct number of nonzero weights.

Finally, we have $\varepsilon/(2N^d) \geq c_3 \cdot \varepsilon^{1+d/\beta}$, for $c_3 = c_3(d,\beta,B)$ so that Remark 2.10 shows that the weights of Ψ_ε are quantised as stated in the theorem. □

A.3 Approximation of horizon functions

We proceed to construct networks that yield good approximations of horizon functions. The underlying idea is relatively straightforward: We have already seen in Lemma A.1 that networks yield approximate realizations of Heaviside functions. Since a horizon function is simply a smoothly transformed Heaviside function, we only need to realize the smooth transformation with a network. This is possible with Theorem A.8. The following lemma makes these arguments rigorous.

Lemma A.9. *For any $\beta > 0$, $d \in \mathbb{N}_{\geq 2}$, and $B > 0$ there exists an absolute constant $c' > 0$, and constants $c = c(d,\beta) > 0$, $s = s(d,\beta,B) \in \mathbb{N}$, and $L = L(d,\beta)$ such that $L \leq c' \cdot (1 + \lceil \log_2(1+\beta) \rceil) \cdot (1 + \beta/d)$ and such that for every function $f \in \mathcal{HF}_{\beta,d,B}$ and every $\varepsilon \in (0, 1/2)$ there is a neural network Φ^f_ε with at most L layers, and at most $c \cdot \varepsilon^{-2(d-1)/\beta}$ non-zero, (s,ε)-quantised weights, such that $\|\mathrm{R}_\varrho(\Phi^f_\varepsilon) - f\|_{L^2([-1/2,1/2]^d)} < \varepsilon$. Moreover, $0 \leq \mathrm{R}_\varrho(\Phi^f_\varepsilon)(x) \leq 1$ for all $x \in [-1/2, 1/2]^d$.*

Proof. Since multiplying A_1 in the definition of a neural network $\Phi = ((A_1,b_1),\dots,(A_L,b_L))$ by a permutation matrix does not change the number of layers or weights, or the possible values of the non-zero weights, we can certainly restrict ourselves to horizon functions $f \in \mathcal{HF}_{\beta,d,B}$ for which the permutation matrix T from Definition 3.3 is the identity matrix. Choose $\gamma \in \mathcal{F}_{\beta,d-1,B}$ such that $f = H \circ \tilde{\gamma}$, where $H = \chi_{[0,\infty) \times \mathbb{R}^{d-1}}$ is the Heaviside function, and where

$$\tilde{\gamma}(x) = (x_1 + \gamma(x_2,\dots,x_d), x_2, \dots, x_d), \quad \text{for } x = (x_1,\dots,x_d) \in [-1/2, 1/2]^d.$$

Theorem A.8 (applied with $p = 1$, with $d-1$ instead of d and with $\varepsilon^2/16$ instead of ε) yields a uniform constant $c' > 0$, and a network Φ^γ_ε with at most $L = L(d,\beta) \leq c' \cdot (1 + \lceil \log_2(1+\beta) \rceil) \cdot (1 + \beta/d)$ layers and at most $c \cdot \varepsilon^{-2(d-1)/\beta}$ non-zero weights (where $c = c(d,\beta,B) \in \mathbb{N}$) such that $\gamma_\varepsilon := \mathrm{R}_\varrho(\Phi^\gamma_\varepsilon)$ approximates γ with an L^1-error of less than $\varepsilon^2/16$. We also recall (by invoking Remark 2.10) that it is possible to construct this network with (s,ε)-quantised weights, for some $s = s(d,\beta,B) \in \mathbb{N}$.

Clearly, one can construct a network $\Phi^{\tilde{\gamma}}_\varepsilon$ of the same complexity (number of non-zero weights, layers, quantization and size of the weights) up to absolute multiplicative constants which satisfies $\mathrm{R}_\varrho(\Phi^{\tilde{\gamma}}_\varepsilon)(x) = (x_1 + \gamma_\varepsilon(x_2,\dots,x_d), x_2,\dots,x_d)$ for all $x \in \mathbb{R}^d$.

As a second step, we invoke Lemma A.1 to obtain a neural network $\Phi^H_{\varepsilon'}$ with two layers and five weights such that $|H(x) - \mathrm{R}_\varrho(\Phi^H_{\varepsilon'})(x)| \leq \chi_{0 \leq x_1 \leq \varepsilon'^2/16}(x)$ and $0 \leq R_\varrho(\Phi^H_{\varepsilon'})(x) \leq 1$ for all $x \in \mathbb{R}^d$. We choose $\varepsilon' \in 2^{-\mathbb{N}}$ so that $\varepsilon/2 \leq \varepsilon' \leq \varepsilon$. With this choice, Lemma A.1 shows that all weights of $\Phi^H_{\varepsilon'}$ are elements of $[-\varepsilon^{-4}, \varepsilon^{-4}] \cap \mathbb{Z}$.

Remark 2.6 shows that there is an absolute constant $c'' > 0$ and certain constants $\widetilde{c} = \widetilde{c}(d,\beta,B) \in \mathbb{N}$ and $\widetilde{L} = \widetilde{L}(d,\beta) \leq c'' \cdot (1 + \lceil \log_2(1+\beta) \rceil) \cdot (1 + \beta/d)$ such that $\Phi^H_{\varepsilon'} \odot \Phi^{\tilde{\gamma}}_\varepsilon$ is a neural network with at most $\widetilde{L}$ layers, and not more than $\widetilde{c} \cdot \varepsilon^{-2(d-1)/\beta}$ non-zero weights, all of which are elements of $[-\varepsilon^{-s'}, \varepsilon^{-s'}] \cap 2^{-s' \lceil \log_2(1/\varepsilon) \rceil}$ for a suitable $s' = s'(d,\beta,B) \in \mathbb{N}$. Furthermore, we have $0 \leq \mathrm{R}_\varrho(\Phi^H_{\varepsilon'} \odot \Phi^{\tilde{\gamma}}_\varepsilon) \leq 1$, since

$0 \le \mathrm{R}_\varrho(\Phi^H_{\varepsilon'}) \le 1$. Thus, to complete the proof, it remains to show that $\mathrm{R}_\varrho(\Phi^H_{\varepsilon'} \odot \Phi^{\tilde\gamma}_\varepsilon)$ indeed approximates $f = H \circ \tilde\gamma$ with an L^2-error of at most ε.

To this end, we estimate

$$\begin{aligned}\|H \circ \tilde\gamma - \mathrm{R}_\varrho(\Phi^H_{\varepsilon'} \odot \Phi^{\tilde\gamma}_\varepsilon)\|_{L^2} &= \|H \circ \tilde\gamma - \mathrm{R}_\varrho(\Phi^H_{\varepsilon'}) \circ \mathrm{R}_\varrho(\Phi^{\tilde\gamma}_\varepsilon)\|_{L^2} \\ &\le \|H \circ \tilde\gamma - H \circ \mathrm{R}_\varrho(\Phi^{\tilde\gamma}_\varepsilon)\|_{L^2} + \|H \circ \mathrm{R}_\varrho(\Phi^{\tilde\gamma}_\varepsilon) - \mathrm{R}_\varrho(\Phi^H_{\varepsilon'}) \circ \mathrm{R}_\varrho(\Phi^{\tilde\gamma}_\varepsilon)\|_{L^2} =: \mathrm{I} + \mathrm{II}.\end{aligned}$$

We continue with the term I. Here we use the shorthand notation $\chi_{\tilde\gamma_1 > 0}$ for the indicator function of the set $\{x \in [-1/2, 1/2]^d : \tilde\gamma_1(x) > 0\}$ and variations thereof. Moreover, we denote by $\mathrm{R}_\varrho(\Phi^{\tilde\gamma}_\varepsilon)_1$ the first coordinate of the $\mathbb{R}^d$-valued function $\mathrm{R}_\varrho(\Phi^{\tilde\gamma}_\varepsilon)$. Recall from our choice of $\Phi^{\tilde\gamma}_\varepsilon$ that $\mathrm{R}_\varrho(\Phi^{\tilde\gamma}_\varepsilon)_1(x) = x_1 + \gamma_\varepsilon(x_2, \dots, x_d)$ for all $x \in [-1/2, 1/2]^d$. Having set the notation, we estimate

$$\begin{aligned}\mathrm{I}^2 &= \|H \circ \tilde\gamma - H \circ \mathrm{R}_\varrho(\Phi^{\tilde\gamma}_\varepsilon)\|^2_{L^2} = \int_{[-\frac12,\frac12]^d} |\chi_{\tilde\gamma_1 \ge 0}(x) - \chi_{\mathrm{R}_\varrho(\Phi^{\tilde\gamma}_\varepsilon)_1 \ge 0}(x)|^2 dx \\ &= \int_{[-\frac12,\frac12]^{d-1}} \int_{-\frac12}^{\frac12} \chi_{\tilde\gamma_1 \ge 0, \mathrm{R}_\varrho(\Phi^{\tilde\gamma}_\varepsilon)_1 < 0}(x_1, \dots, x_d) + \chi_{\tilde\gamma_1 < 0, \mathrm{R}_\varrho(\Phi^{\tilde\gamma}_\varepsilon)_1 \ge 0}(x_1, \dots, x_d) dx_1\, d(x_2, \dots, x_d).\end{aligned}$$

Now, we observe for fixed $(x_2, \dots, x_d) \in [-1/2, 1/2]^{d-1}$ the following equivalence:

$$\begin{aligned}\chi_{\tilde\gamma_1 \ge 0, \mathrm{R}_\varrho(\Phi^{\tilde\gamma}_\varepsilon)_1 < 0}(x) = 1 &\iff x_1 + \gamma(x_2, \dots, x_d) \ge 0 \text{ and } x_1 + \gamma_\varepsilon(x_2, \dots, x_d) < 0 \\ &\iff x_1 \in [-\gamma(x_2, \dots, x_d), -\gamma_\varepsilon(x_2, \dots, x_d)).\end{aligned}$$

This implies

$$\int_{-\frac12}^{\frac12} \chi_{\tilde\gamma_1 \ge 0, \mathrm{R}_\varrho(\Phi^{\tilde\gamma}_\varepsilon)_1 < 0}(x_1, \dots, x_d) dx_1 \le \max\{0, \gamma(x_2, \dots, x_d) - \gamma_\varepsilon(x_2, \dots, x_d)\}.$$

By the same reasoning, $\int_{-1/2}^{1/2} \chi_{\tilde\gamma_1 < 0, \mathrm{R}_\varrho(\Phi^{\tilde\gamma}_\varepsilon)_1 \ge 0}(x_1, \dots, x_d) dx_1 \le \max\{0, \gamma_\varepsilon(x_2, \dots, x_d) - \gamma(x_2, \dots, x_d)\}$. In total, we get because of $\max\{0, x\} + \max\{0, -x\} = |x|$ that

$$\begin{aligned}\mathrm{I}^2 &= \int_{[-\frac12,\frac12]^{d-1}} \int_{-\frac12}^{\frac12} \chi_{\tilde\gamma_1 \ge 0, \mathrm{R}_\varrho(\Phi^{\tilde\gamma}_\varepsilon)_1 < 0}(x_1, \dots, x_d) + \chi_{\tilde\gamma_1 < 0, \mathrm{R}_\varrho(\Phi^{\tilde\gamma}_\varepsilon)_1 \ge 0}(x_1, \dots, x_d) dx_1\, d(x_2, \dots, x_d) \\ &\le \int_{[-\frac12,\frac12]^{d-1}} \max\{0, \gamma(x_2, \dots, x_d) - \gamma_\varepsilon(x_2, \dots, x_d)\} + \max\{0, \gamma_\varepsilon(x_2, \dots, x_d) - \gamma(x_2, \dots, x_d)\} d(x_2, \dots, x_d) \\ &= \|\gamma - \gamma_\varepsilon\|_{L^1([-\frac12,\frac12]^{d-1})} \le \frac{\varepsilon^2}{16},\end{aligned}$$

and hence $\mathrm{I} \le \varepsilon/4$.

We proceed with the term II and recall that $|H(x) - \mathrm{R}_\varrho(\Phi^H_{\varepsilon'})(x)| \le \chi_{0 \le x_1 \le (\varepsilon')^2/16}(x) \le \chi_{0 \le x_1 \le \varepsilon^2/16}(x)$ for all $x \in \mathbb{R}^d$. Therefore,

$$\begin{aligned}\mathrm{II}^2 &= \|H \circ \mathrm{R}_\varrho(\Phi^{\tilde\gamma}_\varepsilon) - \mathrm{R}_\varrho(\Phi^H_{\varepsilon'}) \circ \mathrm{R}_\varrho(\Phi^{\tilde\gamma}_\varepsilon)\|^2_{L^2} \le \int_{[-\frac12,\frac12]^d} \chi_{0 \le \mathrm{R}_\varrho(\Phi^{\tilde\gamma}_\varepsilon)_1 \le \frac{\varepsilon^2}{16}}(x) dx \\ &= \int_{[-\frac12,\frac12]^{d-1}} \int_{-\frac12}^{\frac12} \chi_{0 \le x_1 + \gamma_\varepsilon(x_2, \dots, x_d) \le \frac{\varepsilon^2}{16}} dx_1 d(x_2, \dots, x_d) \le \int_{[-\frac12,\frac12]^{d-1}} \frac{\varepsilon^2}{16} d(x_2, \dots, x_d) = \frac{\varepsilon^2}{16}.\end{aligned}$$

In conclusion, we obtain

$$\|H \circ \tilde\gamma - \mathrm{R}_\varrho(\Phi^H_{\varepsilon'} \odot \Phi^{\tilde\gamma}_\varepsilon)\|_{L^2} \le \mathrm{I} + \mathrm{II} \le \frac{\varepsilon}{4} + \frac{\varepsilon}{4} < \varepsilon. \qquad \square$$

A.4 Approximation of piecewise constant functions

Since for $K \in \mathcal{K}_{r,\beta,d,B}$ we have that χ_K is locally a horizon function, it is straightforward to use Lemma A.9 to construct neural networks approximately realizing such functions.

Theorem A.10. *Let $r \in \mathbb{N}$, $d \in \mathbb{N}_{\ge 2}$ and $\beta, B > 0$ be arbitrary. Then there is an absolute constant $c' > 0$ and there are constants $c = c(d, \beta, r, B) > 0$, $s = s(d, \beta, r, B) \in \mathbb{N}$, and $L = L(\beta, d) \in \mathbb{N}$ with $L \le c' \cdot (1 + \lceil \log_2(1 + \beta) \rceil) \cdot (1 + \beta/d)$ and such that for all $\varepsilon \in (0, 1/2)$ and arbitrary $K \in \mathcal{K}_{r,\beta,d,B}$ there exists a neural network Φ^K_ε with at most L layers and at most $c \cdot \varepsilon^{-2(d-1)/\beta}$ non-zero, (s, ε)-quantised weights such that*

$$\|\mathrm{R}_\varrho(\Phi^K_\varepsilon) - \chi_K\|_{L^2([-\frac12,\frac12]^d)} \le \varepsilon.$$

Proof. For $\lambda = (\lambda_1, \dots, \lambda_d) \in \{1, \dots, 2^r\}^d$, define

$$I_\lambda := \prod_{i \in \{1,\dots,d\}} \left[(\lambda_i - 1)2^{-r} - \frac{1}{2}, \lambda_i 2^{-r} - \frac{1}{2} \right].$$

We have by construction (with disjointness up to null sets) that

$$\dot{\bigcup_{\lambda \in \{1,\dots,2^r\}^d}} I_\lambda = \left[-\frac{1}{2}, \frac{1}{2} \right]^d \quad \text{and} \quad I_\lambda \subset B_{2^{-r}}^{\|\cdot\|_{\ell^\infty}}(x) \text{ for all } x \in I_\lambda.$$

As a consequence of the definition of $\mathcal{K}_{r,\beta,d,B}$, we conclude that $\chi_{I_\lambda}\chi_K = \chi_{I_\lambda} f_\lambda$ for a suitable horizon function $f_\lambda \in \mathcal{HF}_{\beta,d,B}$ and arbitrary $\lambda \in \{1, \dots, 2^r\}^d$.

Now, for each $\lambda \in \{1, \dots, 2^r\}^d$, Lemma A.9 yields a neural network Φ_ε^λ such that

$$\|\mathrm{R}_\varrho(\Phi_\varepsilon^\lambda) - f_\lambda\|_{L^2} \le \frac{\varepsilon}{2^{rd+1}} \quad \text{and such that} \quad 0 \le R_\varrho(\Phi_\varepsilon^\lambda)(x) \le 1 \text{ for } x \in \left[-\frac{1}{2}, \frac{1}{2} \right]^d.$$

By Lemma A.9 and Remark 2.10 there exists $c_1 = c_1(d, \beta, B, r) > 0, c' > 0$, $s_1 = s_1(d, \beta, B, r) \in \mathbb{N}$, and $L_1 = L_1(d, \beta) \le c' \cdot (1 + \lceil \log_2(1 + \beta) \rceil) \cdot (1 + \beta/d)$, such that Φ_ε^λ has at most L_1 layers and at most $c_1 \cdot \varepsilon^{-2(d-1)/\beta}$ non-zero, (s_1, ε)-quantised weights.

Next, by possibly replacing Φ_ε^λ by $\Phi_{1,L_\lambda}^{\mathrm{Id}} \odot \Phi_\varepsilon^\lambda$ with $\Phi_{1,L_\lambda}^{\mathrm{Id}}$ as in Remark 2.4 and for $L_\lambda = L_1 - L(\Phi_\varepsilon^\lambda)$, we can assume that each network Φ_ε^λ has exactly L_1 layers. Note in view of Remark 2.6 and because of $L_1 = L_1(d, \beta)$ that this will not change the quantisation of the weights, and that the number of weights of Φ_ε^λ is still bounded by $c_1' \cdot \varepsilon^{-2(d-1)/\beta}$ for a suitable constant $c_1' = c_1'(d, \beta, B, r)$. For simplicity, we will write c_1 instead of c_1' in what follows.

Now, write $\{1, \dots, 2^r\}^d = \{\lambda_1, \dots, \lambda_{2^{rd}}\}$, and set

$$\Phi := P(\Phi_\varepsilon^{\lambda_1}, P(\dots, P(\Phi_\varepsilon^{\lambda_{2^{rd}-1}}, \Phi_\varepsilon^{\lambda_{2^{rd}}}) \dots)).$$

Note that Φ has L_1 layers, and at most $2^{rd} \cdot c_1 \cdot \varepsilon^{-2(d-1)/\beta} \le c_2 \cdot \varepsilon^{-2(d-1)/\beta}$ nonzero, (s_1, ε)-quantised weights, for a suitable constant $c_2 = c_2(d, \beta, B, r)$.

Finally, an application of Lemma A.6 with $m = 2^{rd}$ and $B = 1$, with $\varepsilon/2$ instead of ε, and with the intervals I_{λ_ℓ}, $\ell \in \{1, \dots, 2^{rd}\}$ yields a network Ψ satisfying

$$\begin{aligned}
\|R_\varrho(\Psi) - \chi_K\|_{L^2} &\le \left\| R_\varrho(\Psi) - \sum_{\ell=1,\dots,2^{rd}} \chi_{I_{\lambda_\ell}} [R_\varrho(\Phi)]_\ell \right\|_{L^2} + \left\| \sum_{\ell=1,\dots,2^{rd}} \chi_{I_{\lambda_\ell}} \cdot ([R_\varrho(\Phi)]_\ell - f_{\lambda_\ell}) \right\|_{L^2} \\
&\le \frac{\varepsilon}{2} + \sum_{\ell=1}^{2^{rd}} \|R_\varrho(\Phi_\varepsilon^{\lambda_\ell}) - f_{\lambda_\ell}\|_{L^2} \le \varepsilon.
\end{aligned}$$

Here, we used that $\chi_K = \sum_{\ell=1,\dots,2^{rd}} \chi_{I_{\lambda_\ell}} \chi_K = \sum_{\ell=1,\dots,2^{rd}} \chi_{I_{\lambda_\ell}} f_{\lambda_\ell}$, with equality almost everywhere, and that $[R_\varrho(\Phi)]_\ell = R_\varrho(\Phi_\varepsilon^{\lambda_\ell})$, by construction of Φ.

To complete the proof, it remains to verify that $\Phi_\varepsilon^K := \Psi$ has the required complexity. But Lemma A.6 shows that Ψ has at most $6 + L(\Phi) = 6 + L_1$ layers, which is easily seen to satisfy the required bound. Furthermore, the same lemma also shows that the weights of Ψ are $(\max\{s_0, s_1\}, \varepsilon/2^{1+rd})$-quantised, for a constant $s_0 = s_0(d) \in \mathbb{N}$, so that Remark 2.10 shows that Ψ has (s_2, ε)-quantised weights, for a suitable constant $s_2 = s_2(d, \beta, B, r) \in \mathbb{N}$. Finally, the lemma also shows

$$M(\Psi) \le c \cdot (2^{rd} + L_1 + M(\Phi)) \le c_3 \cdot \varepsilon^{-2(d-1)/\beta},$$

for suitable constants $c = c(d) > 0$ and $c_3 = c_3(d, \beta, B, r) > 0$, as desired. □

Theorem A.10 yields an approximation result by neural networks of functions that are piecewise constant. However, a simple extension allows us to also approximate piecewise smooth functions optimally.

Corollary A.11. *Let $r \in \mathbb{N}$, $d \in \mathbb{N}_{\ge 2}$, and $B, \beta > 0$. Then there exist constants $c = c(d, \beta, r, B) > 0$, $s = s(d, \beta, B, r) \in \mathbb{N}$, $c' > 0$, and $L = L(d, \beta) \in \mathbb{N}$ with $L \le c' \cdot (1 + \lceil \log_2(1 + \beta) \rceil) \cdot (1 + \beta/d)$, such that for all $\varepsilon \in (0, 1/2)$ and all $f \in \mathcal{E}_{r,\beta,d,B}$ there exists a neural network Φ_ε^f with at most L layers, and at most $c \cdot \varepsilon^{-2(d-1)/\beta}$ non-zero, (s, ε)-quantised weights, such that*

$$\|\mathrm{R}_\varrho(\Phi_\varepsilon^f) - f\|_{L^2([-\frac{1}{2}, \frac{1}{2}]^d)} \le \varepsilon.$$

Proof. Let $\varepsilon \in (0, 1/2)$ and $f = \chi_K \cdot g$ with $g \in \mathcal{F}_{\beta',d,B}$ and $K \in \mathcal{K}_{r,\beta,d,B}$, where we recall $\beta' = d\beta/2(d-1)$, so that $d/\beta' = 2(d-1)/\beta$. Note because of $d \geq 2$ that $d/2 \leq d-1 \leq d$, and hence $\beta' \leq \beta$, so that $1 + \beta'/d \leq 1 + \beta/d$ and $1 + \lceil \log_2(1+\beta') \rceil \leq 1 + \lceil \log_2(1+\beta) \rceil$. We start by constructing the following three networks:

Theorem A.10 yields certain constants $c_1 = c_1(d,\beta,r,B) > 0$, $c' > 0$, $s_1 = s_1(d,\beta,r,B) \in \mathbb{N}$, and $L_1 = L_1(\beta,d) \leq c' \cdot (1 + \lceil \log_2(1+\beta) \rceil) \cdot (1 + \beta/d)$, and a network Φ_ε^K with no more than L_1 layers and at most $c_1 \cdot \varepsilon^{-2(d-1)/\beta}$ non-zero, (s_1, ε)-quantised weights, such that $0 \leq R_\varrho(\Phi_\varepsilon^K) \leq 1$ and

$$\|\mathrm{R}_\varrho(\Phi_\varepsilon^K) - \chi_K\|_{L^2} \leq \frac{\varepsilon}{3B}.$$

Likewise, since $\beta' = \beta'(\beta,d) \leq \beta$ (see the beginning of the proof), Theorem A.8 yields $c_2 = c_2(d,\beta,B)$, $s_2 = s_2(d,\beta,B) \in \mathbb{N}$, $c'' > 0$, and $L_2 = L_2(\beta,d)$ with

$$L_2 \leq c'' \cdot (1 + \lceil \log_2(1+\beta') \rceil) \cdot (1 + \beta'/d) \leq c'' \cdot (1 + \lceil \log_2(1+\beta) \rceil) \cdot (1 + \beta/d),$$

and a network Φ_ε^g with no more than L_2 layers and at most $c_2 \cdot \varepsilon^{-d/\beta'} = c_2 \cdot \varepsilon^{-2(d-1)/\beta}$ non-zero, (s_2, ε)-quantised weights, such that

$$\|\mathrm{R}_\varrho(\Phi_\varepsilon^g) - g\|_{L^2} \leq \frac{\varepsilon}{3}.$$

Set $B_0 := 2^{\lceil \log_2 \max\{1,B\} \rceil}$, and note $|g| \leq B \leq B_0$. Precisely as in the proof of Theorem A.8 (after Equation (A.4)), we see that by adding an additional layer (with a constant number of quantised weights) to Φ_g^ε, we can (and will) assume $-B_0 \leq R_\varrho(\Phi_g^\varepsilon) \leq B_0$.

Finally, Lemma A.2 (applied with $\theta = 2(d-1)/\beta \geq d/\beta$, with $L_3^{(0)} := \lceil 2\beta/d \rceil$ instead of L, and with $M = B_0$), combined with Remark 2.10 yields constants $c_3 = c_3(d,\beta,B)$, $c''' > 0$, $s_3 = s_3(B)$, and $L_3 = L_3(\beta,d) \leq c''' L_3^{(0)} \leq 2c'''(1 + \log_2(1+\beta))(1 + \beta/d)$, and a network $\tilde{\times}$ with at most L_3 layers and at most $c_3 \cdot \varepsilon^{-\theta} = c_3 \cdot \varepsilon^{-2(d-1)/\beta}$ non-zero, (s_3, ε)-quantised weights such that

$$|xy - \mathrm{R}_\varrho(\tilde{\times})(x,y)| \leq \frac{\varepsilon}{3} \quad \text{for all } x, y \in [-B_0, B_0]\,.$$

As usual, we can assume $L(\Phi_\varepsilon^K) = L(\Phi_\varepsilon^g) = \max\{L_2, L_3\}$, by possibly switching to $\Phi_{1,\lambda_1}^{\mathrm{Id}} \odot \Phi_\varepsilon^K$ and to $\Phi_{1,\lambda_2}^{\mathrm{Id}} \odot \Phi_\varepsilon^g$ for $\lambda_1 = \max\{L_2, L_3\} - L(\Phi_\varepsilon^K)$ and $\lambda_2 = \max\{L_2, L_3\} - L(\Phi_\varepsilon^g)$. This might necessitate changing the constants c'', c''' and c_2, c_3, but these constants stay of the required form.

Now, we set $\Phi_\varepsilon^f := \tilde{\times} \odot P(\Phi_\varepsilon^K, \Phi_\varepsilon^g)$. By Remark 2.6, Φ_ε^f has at most $\max\{L_1, L_2\} + L_3$ layers and $c_4 \cdot \varepsilon^{-2(d-1)/\beta}$ non-zero, $(\max\{s_1, s_2, s_3\}, \varepsilon)$-quantised weights, for a suitable $c_4 = c_4(d,\beta,r,B) > 0$. Since $\beta' \leq \beta$ (see the beginning of the proof) independent of d, there exists an absolute constant $c'''' > 0$ such that

$$\max\{L_1, L_2\} + L_3 \leq c'''' \cdot (1 + \lceil \log_2(1+\beta) \rceil) \cdot (1 + \beta/d).$$

Finally, we show that Φ_ε^f satisfies the claimed error bound. To this end, we estimate

$$\begin{aligned}
\|\mathrm{R}_\varrho(\Phi_\varepsilon^f) - f\|_{L^2} &= \|\mathrm{R}_\varrho(\tilde{\times})(\mathrm{R}_\varrho(\Phi_\varepsilon^K), \mathrm{R}_\varrho(\Phi_\varepsilon^g)) - f\|_{L^2} \\
&\leq \|\mathrm{R}_\varrho(\tilde{\times})(\mathrm{R}_\varrho(\Phi_\varepsilon^K), \mathrm{R}_\varrho(\Phi_\varepsilon^g)) - \mathrm{R}_\varrho(\Phi_\varepsilon^K) \cdot \mathrm{R}_\varrho(\Phi_\varepsilon^g)\|_{L^2} + \|\mathrm{R}_\varrho(\Phi_\varepsilon^K) \cdot \mathrm{R}_\varrho(\Phi_\varepsilon^g) - f\|_{L^2} \\
&\leq \frac{\varepsilon}{3} + \|\mathrm{R}_\varrho(\Phi_\varepsilon^K) \cdot [\mathrm{R}_\varrho(\Phi_\varepsilon^g) - g]\|_{L^2} + \|g \cdot [\mathrm{R}_\varrho(\Phi_\varepsilon^K) - \chi_K]\|_{L^2}.
\end{aligned}$$

We continue by recalling $0 \leq R_\varrho(\Phi_\varepsilon^K) \leq 1$, so that

$$\|\mathrm{R}_\varrho(\Phi_\varepsilon^K) \cdot [\mathrm{R}_\varrho(\Phi_\varepsilon^g) - g]\|_{L^2} \leq \|\mathrm{R}_\varrho(\Phi_\varepsilon^g) - g\|_{L^2} \leq \frac{\varepsilon}{3}. \tag{A.5}$$

Moreover, since $g \in \mathcal{F}_{\beta',d,B}$, so that $\|g\|_{\sup} \leq B$, we also have

$$\|g \cdot [\mathrm{R}_\varrho(\Phi_\varepsilon^K) - \chi_K]\|_{L^2} \leq B \cdot \|\mathrm{R}_\varrho(\Phi_\varepsilon^K) - \chi_K\|_{L^2} \leq \frac{\varepsilon}{3}.$$

Combining all estimates above yields $\|\mathrm{R}_\varrho(\Phi_\varepsilon^f) - f\|_{L^2} \leq \varepsilon$, as desired. □

B Lower bounds for the approximation of horizon functions

In this section, we give the proofs of Theorem 4.2, which establishes a lower bound for approximation uniformly over the class of horizon functions, and of Theorem 4.3, which establishes a similar lower bound for the approximation of a *single* judiciously chosen horizon function f.

Since the proof of the lower bound for the uniform setting is simpler but contains most of the crucial ideas, we begin with this setting. The improvement to a lower bound for the approximation of a single function is then obtained by a suitable application of the Baire category theorem.

B.1 Lower bounds for the uniform setting

The general idea is as follows: In Lemma B.4, we show that if we denote by

$$\mathcal{NN}^{\mathcal{B},\varrho}_{M,K,d} := \{R_\varrho(\Phi) \,:\, \Phi \in \mathcal{NN}^{\mathcal{B}}_{M,K,d}\}$$

the set of all realizations (with activation function ϱ) of networks in $\mathcal{NN}^{\mathcal{B}}_{M,K,d}$, then each function $f \in \Phi \in \mathcal{NN}^{\mathcal{B},\varrho}_{M,K,d}$ can be encoded with $\ell := C \cdot M \cdot (K + \lceil \log_2 M \rceil)$ bits, for a universal constant $C = C(d) \in \mathbb{N}$, i.e., there is an injective map $\Gamma : \mathcal{NN}^{\mathcal{B},\varrho}_{M,K,d} \to \{0,1\}^\ell$, with suitable left inverse $\Theta : \{0,1\}^\ell \to \mathcal{NN}^{\mathcal{B},\varrho}_{M,K,d}$. Thus, if to a given $\varepsilon > 0$, there is for each $f \in \mathcal{HF}_{\beta,d,B}$ a neural network $\Phi_{f,\varepsilon} \in \mathcal{NN}^{\mathcal{B}}_{M,K,d}$ with $\|f - \mathrm{R}_\varrho(\Phi_{f,\varepsilon})\|_{L^2} \le \varepsilon$, then the *encoder-decoder pair* (E^ℓ, D^ℓ) defined by

$$\begin{aligned} E^\ell &: \mathcal{HF}_{\beta,d,B} \to \{0,1\}^\ell, \qquad f \mapsto \Gamma\left(R_\varrho(\Phi_{f,\varepsilon})\right), \\ D^\ell &: \{0,1\}^\ell \to L^2\left([-1/2,\, 1/2]^d\right),\; c \mapsto \Theta(c) \end{aligned}$$

achieves distortion ε, i.e., it satisfies

$$\sup_{f \in \mathcal{HF}_{\beta,d,B}} \|f - D^\ell(E^\ell(f))\|_{L^2} \le \varepsilon.$$

From this, we obtain the desired lower bound by showing that each encoder-decoder pair (E^ℓ, D^ℓ) for $\mathcal{HF}_{\beta,d,B}$ which achieves distortion ε necessarily has to satisfy $\ell \gtrsim \varepsilon^{-2(d-1)/\beta}$.

Of course, this last statement is highly nontrivial; it is essentially a lower bound on the *description complexity* of the class $\mathcal{HF}_{\beta,d,B}$. As we will see, this description complexity—which is expressed using encoder-decoder pairs—is closely related to the asymptotic behavior of the so-called *entropy numbers* of the class $\mathcal{HF}_{\beta,d,B}$.

Deriving a lower bound for these entropy numbers from first principles would be quite difficult. But luckily, we can use a trick to transfer known results from [13] about the entropy numbers of the class $C^{0,\beta}([-1/2, 1/2]^{d-1})$ to bounds on the entropy numbers of the class of horizon functions. This trick for transferring the entropy bounds from $C^{0,\beta}([-1/2, 1/2]^{d-1})$ to the class of horizon functions is explained by the following lemma.

Lemma B.1. *For $d \in \mathbb{N}_{\ge 2}$, and an arbitrary Borel measurable function $\gamma : [-1/2, 1/2]^{d-1} \to \mathbb{R}$, define*

$$\mathrm{HF}_\gamma : \left[-\frac{1}{2}, \frac{1}{2}\right]^d \to \{0,1\}, \quad (x_1, x_2, \dots, x_d) \mapsto H(x_1 + \gamma(x_2, \dots, x_d), x_2, \dots, x_d),$$

where $H = \chi_{[0,\infty)\times\mathbb{R}^{d-1}}$ denotes the Heaviside function. Then, we have for arbitrary $p \in (0, \infty)$ and arbitrary measurable $\psi, \gamma : [-1/2, 1/2]^{d-1} \to [-1/2, 1/2]$ the identity

$$\|\mathrm{HF}_\gamma - \mathrm{HF}_\psi\|_{L^p([-\frac{1}{2},\frac{1}{2}]^d)} = \|\gamma - \psi\|^{\frac{1}{p}}_{L^1([-\frac{1}{2},\frac{1}{2}]^{d-1})}.$$

If $\psi, \gamma : [-1/2, 1/2]^{d-1} \to \mathbb{R}$ are measurable, we still have $\|\mathrm{HF}_\gamma - \mathrm{HF}_\psi\|_{L^p([-1/2,1/2]^d)} \le \|\gamma - \psi\|^{1/p}_{L^1([-1/2,1/2]^{d-1})}$.

Proof. For $x = (x_1, \dots, x_d) \in \mathbb{R}^d$, we write $\hat{x} := (x_2, \dots, x_d)$. Then, we have the following equivalence:

$$\mathrm{HF}_\gamma(x) = 1 \iff x_1 + \gamma(\hat{x}) \ge 0.$$

Thus, $|\mathrm{HF}_\gamma - \mathrm{HF}_\psi|$ is $\{0,1\}$-valued with

$$\begin{aligned} & |\mathrm{HF}_\gamma(x) - \mathrm{HF}_\psi(x)| = 1 \\ \iff & [x_1 + \gamma(\hat{x}) \ge 0 \text{ and } x_1 + \psi(\hat{x}) < 0] \text{ or } [x_1 + \gamma(\hat{x}) < 0 \text{ and } x_1 + \psi(\hat{x}) \ge 0] \\ \iff & x_1 \in [-\gamma(\hat{x}), -\psi(\hat{x})) \text{ or } x_1 \in [-\psi(\hat{x}), -\gamma(\hat{x})). \end{aligned}$$

But since we have $[-\gamma(\hat{x}), -\psi(\hat{x})) \cap [-\psi(\hat{x}), -\gamma(\hat{x})) \subset [-\gamma(\hat{x}), -\gamma(\hat{x})) = \varnothing$, and since γ, ψ only take values in $[-1/2, 1/2]$, so that $[-\gamma(\hat{x}), -\psi(\hat{x})) \cup [-\psi(\hat{x}), -\gamma(\hat{x})) \subset [-1/2, 1/2]$, we get with the one-dimensional Lebesgue measure μ for each $\hat{x} \in [-1/2, 1/2]^{d-1}$ that

$$\begin{aligned} & \mu(\{x_1 \in [-1/2, 1/2] \,:\, |\mathrm{HF}_\gamma(x_1, \hat{x}) - \mathrm{HF}_\psi(x_1, \hat{x})| = 1\}) \\ & = \mu([-\gamma(\hat{x}), -\psi(\hat{x}))) + \mu([-\psi(\hat{x}), -\gamma(\hat{x}))) \\ & = \max\{0, \gamma(\hat{x}) - \psi(\hat{x})\} + \max\{0, \psi(\hat{x}) - \gamma(\hat{x})\} \\ & = |\gamma(\hat{x}) - \psi(\hat{x})|. \end{aligned} \tag{B.1}$$

Since $|\mathrm{HF}_\gamma - \mathrm{HF}_\psi|$ is $\{0,1\}$-valued, this implies by Fubini's theorem

$$\begin{aligned}
\|\mathrm{HF}_\gamma - \mathrm{HF}_\psi\|_{L^p([-\frac12,\frac12]^d)}^p &= \int_{[-\frac12,\frac12]^{d-1}} \int_{-\frac12}^{\frac12} |\mathrm{HF}_\gamma(x_1,\hat{x}) - \mathrm{HF}_\psi(x_1,\hat{x})|^p \, dx_1 \, d\hat{x} \\
&= \int_{[-\frac12,\frac12]^{d-1}} \mu\left(\left\{x_1 \in [-\frac12, \frac12] \,:\, |\mathrm{HF}_\gamma(x_1,\hat{x}) - \mathrm{HF}_\psi(x_1,\hat{x})| = 1\right\}\right) d\hat{x} \\
&= \int_{[-\frac12,\frac12]^{d-1}} |\gamma(\hat{x}) - \psi(\hat{x})| \, d\hat{x} = \|\gamma - \psi\|_{L^1([-\frac12,\frac12]^{d-1})},
\end{aligned} \tag{B.2}$$

as claimed.

If we have $\psi, \gamma : [-1/2, 1/2]^{d-1} \to \mathbb{R}$ instead of $\psi, \gamma : [-1/2, 1/2]^{d-1} \to [-1/2, 1/2]$, then the equality in (B.1)—and thus also the one in (B.2)—need to be replaced by "$\le$", but the remainder of the proof remains valid. ◻

Our next goal (see Lemma B.3) is to show that an ℓ-bit encoder-decoder pair (E^ℓ, D^ℓ) which achieves distortion ε over the class $\mathcal{HF}_{\beta,d,B}$ needs to satisfy $\ell \gtrsim \varepsilon^{-2(d-1)/\beta}$. Before we prove this, let us fix some notation and terminology:

Definition B.2. *Let $\Omega \subset \mathbb{R}^d$ be measurable, and let $\mathcal{C} \subset L^2(\Omega)$ be an arbitrary function class. For each $\ell \in \mathbb{N}$, we denote by*

$$\mathfrak{E}^\ell := \{E : \mathcal{C} \to \{0,1\}^\ell\}$$

the set of binary encoders mapping elements of $\mathcal{C}$ to bit-strings of length ℓ, *and we let*

$$\mathfrak{D}^\ell := \{D : \{0,1\}^\ell \to L^2(\Omega)\}$$

be the set of binary decoders mapping bit-strings of length ℓ to elements of $L^2(\Omega)$.

An encoder-decoder pair $(E^\ell, D^\ell) \in \mathfrak{E}^\ell \times \mathfrak{D}^\ell$ is said to achieve distortion $\varepsilon > 0$ over the function class $\mathcal{C}$, *if*

$$\sup_{f \in \mathcal{C}} \|D^\ell(E^\ell(f)) - f\|_{L^2(\Omega)} \le \varepsilon.$$

Finally, for $\varepsilon > 0$ the minimax code length $L(\varepsilon, \mathcal{C})$ *is*

$$L(\varepsilon, \mathcal{C}) := \min\left\{\ell \in \mathbb{N} \,:\, \exists (E^\ell, D^\ell) \in \mathfrak{E}^\ell \times \mathfrak{D}^\ell : \sup_{f \in \mathcal{C}} \|D^\ell(E^\ell(f)) - f\|_{L^2(\Omega)} \le \varepsilon\right\},$$

with the interpretation $L(\varepsilon, \mathcal{C}) = \infty$ if $\sup_{f \in \mathcal{C}} \|D^\ell(E^\ell(f)) - f\|_{L^2(\Omega)} > \varepsilon$ for all $(E^\ell, D^\ell) \in \mathfrak{E}^\ell \times \mathfrak{D}^\ell$ and arbitrary $\ell \in \mathbb{N}$.

Now that we have fixed the terminology, we derive a lower bound on the asymptotic behavior of the minimax code length for the class $\mathcal{HF}_{\beta,d,B}$ of horizon functions, by using Lemma B.1 to transfer results about the behavior of the entropy numbers of $C^{0,\beta}([0,1]^{d-1})$ to the class $\mathcal{HF}_{\beta,d,B}$. We remark that this result is essentially folklore; see for example [12, 11] for related, but less detailed proofs; in fact, our proof is based on those two papers.

Lemma B.3. *Let $d \in \mathbb{N}_{\ge 2}$, and $\beta, B > 0$ be arbitrary. Then there are constants $C = C(d, \beta, B) > 0$ and $\varepsilon_0 = \varepsilon_0(d, \beta, B) > 0$, such that for each $\varepsilon \in (0, \varepsilon_0)$, the minimax code length $L(\varepsilon, \mathcal{HF}_{\beta,d,B})$ of the class $\mathcal{HF}_{\beta,d,B}$ of horizon functions satisfies*

$$L(\varepsilon, \mathcal{HF}_{\beta,d,B}) \ge C \cdot \varepsilon^{-\frac{2(d-1)}{\beta}}.$$

Proof. **Step 1**: We prove that there are constants $C_1 = C_1(d, \beta, B) > 0$ and $\varepsilon_1 = \varepsilon_1(d, \beta, B) > 0$ such that for each $\varepsilon \in (0, \varepsilon_1)$, there is some $N \ge \exp\left(C_1 \cdot \varepsilon^{-(d-1)/\beta}\right)$, and functions $f_1, \dots, f_N \in \mathcal{F}_{\beta,d-1,B}$ satisfying $\|f_i - f_\ell\|_{L^1} \ge \varepsilon$ for $i \ne \ell$.

To show this, we need some preparation: First, let us write $\beta = n + \sigma$ with $n \in \mathbb{N}_0$ and $\sigma \in (0,1]$. It is easy to see from Lemma D.1 (by translating everything from $[0,1]^{d-1}$ to $[-1/2, 1/2]^{d-1}$) that there is a constant $C_2 = C_2(d, \beta) > 0$ such that each $u \in C^n\left([-1/2, 1/2]^{d-1}\right)$ satisfies

$$\|u\|_{C^{0,\beta}} \le C_2 \cdot \left(\|u\|_{\sup} + \max_{|\alpha| = n} \mathrm{Lip}_\sigma(\partial^\alpha u)\right). \tag{B.3}$$

Let $C_3 := B / (1 + 2C_2)$, and set

$$F_\beta^{d-1}(C_3) := \left\{ u \in C^n \left([-1/2, 1/2]^{d-1} \right) \,:\, \|u\|_{\sup} \le C_3 \text{ and } \max_{|\alpha| = n} \mathrm{Lip}_\sigma (\partial^\alpha u) \le C_3 \right\},$$

as in [13]. Actually, in [13], the unit cube $[0,1]^{d-1}$ is used instead of $[-1/2, 1/2]^{d-1}$, but it is easy to see (by translation) that this makes no difference for what follows. Precisely, we want to use [13, Theorem 3], which ensures existence of a large number of functions $f_1, \dots, f_N \in F_\beta^{d-1}(C_3)$ with $\|f_i - f_\ell\|_{L^1} \ge \varepsilon$ for $i \ne \ell$. To see that this indeed follows from [13, Theorem 3], we recall a few notions from [13, Page 1086]: For a subset $U \subset X$ of a metric space (X, d), we say that U is **ε-distinguishable** if $d(x, y) \ge \varepsilon$ for all $x, y \in U$ with $x \ne y$. Next, for $\varnothing \ne A \subset X$, we define $M_\varepsilon(A) := \max \{ |U| \,:\, U \subset A \text{ is } \varepsilon\text{-distinguishable} \}$, and we define the **capacity** of A as[†] $C_\varepsilon(A) = \ln M_\varepsilon(A)$. Additionally, there is also the notion of the **(metric) entropy** $H_\varepsilon(A)$ of A, the precise definition of which is immaterial for us; the only property of the entropy that we will need is that $C_\varepsilon(A) \ge H_\varepsilon(A)$.

Finally, [13, Theorem 3] shows that if we consider $A = F_\beta^{d-1}(C_3)$ as a subset of the metric space $X = L^1([-1/2, 1/2]^{d-1})$, then the entropy of $F_\beta^{d-1}(C_3)$ satisfies $H_\varepsilon(F_\beta^{d-1}(C_3)) \ge C_1 \cdot \varepsilon^{-(d-1)/\beta}$ for $\varepsilon \in (0, \varepsilon_1)$ and certain constants $C_1 = C_1(d, \beta, C_3) = C_1(d, \beta, B) > 0$ and $\varepsilon_1 = \varepsilon_1(d, \beta, C_3) = \varepsilon_1(d, \beta, B) > 0$. Because of $\ln M_\varepsilon(A) = C_\varepsilon(A) \ge H_\varepsilon(A)$, and by definition of $M_\varepsilon(A)$, this implies that there is some $N \ge \exp\left(C_1 \cdot \varepsilon^{-(d-1)/\beta}\right)$ and certain functions $f_1, \dots, f_N \in F_\beta^{d-1}(C_3)$ with $\|f_i - f_\ell\|_{L^1} \ge \varepsilon$ for $i \ne \ell$. To complete the proof of Step 1, we observe as a consequence of Equation (B.3) that each $f_i \in F_\beta^{d-1}(C_3)$ satisfies $f_i \in C^n([-1/2, 1/2]^{d-1})$, with

$$\|f_i\|_{C^{0,\beta}} \le C_2 \cdot \left(\|f_i\|_{\sup} + \max_{|\alpha| = n} \mathrm{Lip}_\sigma(\partial^\alpha f_i) \right) \le C_2 \cdot 2C_3 \le B,$$

i.e., $f_i \in \mathcal{F}_{\beta, d-1, B}$.

Step 2: For simplicity, let $B_0 := \min\{1/2, B\}$. Furthermore, for $x \in [-1/2, 1/2]^d$, let us write $x = (x_1, \hat{x})$, with $x_1 \in [-1/2, 1/2]$ and $\hat{x} \in [-1/2, 1/2]^{d-1}$. Finally, recall from Lemma B.1 that to every measurable function $\gamma : [-1/2, 1/2]^{d-1} \to \mathbb{R}$, we associate the function

$$\mathrm{HF}_\gamma : [-1/2,\ 1/2]^d \to \{0, 1\}, (x_1, \hat{x}) \mapsto H(x_1 + \gamma(\hat{x}), \hat{x}).$$

Now, each $\gamma \in \mathcal{F}_{\beta, d-1, B_0}$ satisfies $\|\gamma\|_{L^\infty} \le \|\gamma\|_{C^{0,\beta}} \le B_0 \le 1/2$, and thus $\gamma : [-1/2, 1/2]^{d-1} \to [-1/2, 1/2]$. Therefore, Lemma B.1 shows

$$\|\mathrm{HF}_\gamma - \mathrm{HF}_\psi\|_{L^2([-1/2, 1/2]^d)} \ge \|\gamma - \psi\|_{L^1([-1/2, 1/2]^{d-1})}^{\frac{1}{2}} \qquad \text{for all } \gamma, \psi \in \mathcal{F}_{\beta, d-1, B_0}. \tag{B.4}$$

Finally, we remark that directly from the definition, we have $\mathrm{HF}_\gamma \in \mathcal{HF}_{\beta, d, B_0} \subset \mathcal{HF}_{\beta, d, B}$ for all $\gamma \in \mathcal{F}_{\beta, d-1, B_0}$.

Step 3: In this step, we actually prove the claim: Step 1 (applied with $B_0 = \min\{1/2, B\}$ instead of B and with $8\varepsilon^2$ instead of ε) yields constants $C_1 = C_1(d, \beta, B) > 0$ and $\varepsilon_0 = \varepsilon_0(d, \beta, B) > 0$, such that for $\varepsilon \in (0, \varepsilon_0)$, there is some $N \ge \exp\left(C_1 \cdot \left(8\varepsilon^2\right)^{-(d-1)/\beta}\right)$ and $f_1, \dots, f_N \in \mathcal{F}_{\beta, d-1, B_0} \subset \mathcal{F}_{\beta, d-1, B}$ with $\|f_i - f_\ell\|_{L^1} \ge 8\varepsilon^2$ for $i \ne \ell$. With this constant C_1, we will show

$$L(\varepsilon, \mathcal{HF}_{\beta, d, B}) \ge \frac{C_1}{8^{(d-1)/\beta}} \cdot \varepsilon^{-\frac{2(d-1)}{\beta}}, \qquad \text{for all } \varepsilon \in (0, \varepsilon_0),$$

which clearly implies the claim.

For the proof, let $E^\ell : \mathcal{HF}_{\beta, d, B} \to \{0, 1\}^\ell$ and $D^\ell : \{0, 1\}^\ell \to L^2([-1/2, 1/2]^d)$ be any encoder-decoder pair which achieves distortion $\varepsilon \in (0, \varepsilon_0)$ over the class $\mathcal{HF}_{\beta, d, B}$. We need to show

$$\ell \ge \frac{C_1}{8^{(d-1)/\beta}} \cdot \varepsilon^{-\frac{2(d-1)}{\beta}}.$$

Assume towards a contradiction that this fails. Thus, $|\{0, 1\}^\ell| = 2^\ell \le e^\ell < \exp\left(C_1 \cdot \left(8\varepsilon^2\right)^{-(d-1)/\beta}\right)$. By the pigeonhole principle, with $f_1, \dots, f_N$ as above, this ensures existence of $i, j \in \{1, \dots, N\}$ with

[†] We remark that some authors use a logarithm with a different basis than the natural logarithm. For us this does not matter, since we will obtain a bound $C_\varepsilon(A) \ge C \cdot \varepsilon^{-(d-1)/\beta}$, so that a different choice of basis just leads to a different constant C.

$i \neq j$, but with $E^\ell(\mathrm{HF}_{f_i}) = E^\ell(\mathrm{HF}_{f_j})$. But by Step 2 (Equation (B.4)), this entails

$$
\begin{aligned}
&2\sqrt{2}\varepsilon = \sqrt{8\varepsilon^2} \leq \sqrt{\|f_i - f_j\|_{L^1}} \leq \|\mathrm{HF}_{f_i} - \mathrm{HF}_{f_j}\|_{L^2} \\
&\leq \|\mathrm{HF}_{f_i} - D^\ell(E^\ell(\mathrm{HF}_{f_i}))\|_{L^2} + \|D^\ell(E^\ell(\mathrm{HF}_{f_i})) - D^\ell(E^\ell(\mathrm{HF}_{f_j}))\|_{L^2} + \|D^\ell(E^\ell(\mathrm{HF}_{f_j})) - \mathrm{HF}_{f_j}\|_{L^2} \\
&\leq \varepsilon + 0 + \varepsilon = 2\varepsilon,
\end{aligned}
$$

a contradiction. Here, we used in the last step that $E^\ell(\mathrm{HF}_{f_i}) = E^\ell(\mathrm{HF}_{f_j})$, and that the pair (E^ℓ, D^ℓ) achieves distortion ε over $\mathcal{HF}_{\beta,d,B} \supset \{\mathrm{HF}_{f_1}, \dots, \mathrm{HF}_{f_N}\}$. This contradiction completes the proof. □

Now that we have a lower bound on the minimax code length of the class of horizon functions, the next step of the program that was outlined at the beginning of this subsection is to show that if each horizon function $f \in \mathcal{HF}_{\beta,d,B}$ can be approximated with error $\leq \varepsilon$ by a neural network of bounded complexity, then this yields an encoder-decoder pair for the class $\mathcal{HF}_{\beta,d,B}$ of a certain (small) bit-length ℓ. To main idea for showing this is to encode the approximating neural networks as bit-strings. Our next lemma shows that this is possible.

Lemma B.4. *Let $d \in \mathbb{N}$, and let $\mathcal{B}$ be an encoding scheme for real numbers. For $M, K \in \mathbb{N}$, let $\mathcal{NN}^{\mathcal{B}}_{M,K,d}$ be as in Definition 4.1. Let $\varrho : \mathbb{R} \to \mathbb{R}$ with $\varrho(0) = 0$, and define*

$$\mathcal{NN}^{\mathcal{B},\varrho}_{M,K,d} := \{\mathrm{R}_\varrho(\Phi) \,:\, \Phi \in \mathcal{NN}^{\mathcal{B}}_{M,K,d}\}.$$

There is a universal constant $C = C(d) \in \mathbb{N}$, such that for arbitrary $M, K \in \mathbb{N}$, there is an injective map $\Gamma^{\mathcal{B},\varrho}_{M,K,d} : \mathcal{NN}^{\mathcal{B},\varrho}_{M,K,d} \to \{0,1\}^{CM(K+\lceil \log_2 M\rceil)}$.

Proof. The proof is similar to that of [6, Theorem 2.7]. However, since we define networks slightly differently in this work, we repeat the main points of the proof with some simplifications.

In Lemma E.1, it is shown that for each $f \in \mathcal{NN}^{\mathcal{B},\varrho}_{M,K,d}$, there is a neural network $\Phi_f \in \mathcal{NN}^{\mathcal{B}}_{M,K,d}$ satisfying $f = \mathrm{R}_\varrho(\Phi_f)$ and furthermore $N(\Phi_f) \leq M(\Phi_f) + d + 1$.

Therefore, it suffices to show for

$$\mathcal{NN}^*_{M,K} := \{\Phi \in \mathcal{NN}^{\mathcal{B}}_{M,K,d} \,:\, N(\Phi) \leq M(\Phi) + d + 1\}$$

and $\ell := C \cdot M \cdot (K + \lceil \log_2 M\rceil)$ (with a suitable constant $C = C(d) \in \mathbb{N}$) that there is an injective map $\Theta^{\mathcal{B}}_{M,K} : \mathcal{NN}^*_{M,K} \to \{0,1\}^\ell$, since then the map $\Gamma^{\mathcal{B},\varrho}_{M,K} : \mathcal{NN}^{\mathcal{B},\varrho}_{M,K,d} \to \{0,1\}^\ell, f \mapsto \Theta^{\mathcal{B}}_{M,K}(\Phi_f)$ is easily seen to be injective.

To prove existence of $\Theta^{\mathcal{B}}_{M,K}$, we show that each $\Phi \in \mathcal{NN}^*_{M,K}$ can be encoded (in a uniquely decodable way) with ℓ bits. To show this, we first observe that each such Φ satisfies for $L := L(\Phi)$ the estimates

$$L = \sum_{\ell=1}^{L} 1 \leq \sum_{\ell=1}^{L} N_\ell = N(\Phi) - d \leq M(\Phi) + 1 \leq M + 1, \text{ and } N(\Phi) \leq M(\Phi) + d + 1 \leq M + d + 1 \leq 3d \cdot M =: T.$$

Next, in the notation of Definition 2.1, we can write $\Phi = ((A_1, b_1), \dots, (A_L, b_L))$, so that it suffices to encode (in a uniquely decodable way) the integer $L \in \mathbb{N}$, the matrices $A_1, \dots, A_L$ and the vectors $b_1, \dots, b_L$ using a bit-string of length ℓ. To show this, let $\mathcal{B} = (B_n)_{n\in\mathbb{N}}$.

Now, if $A \in \mathbb{R}^{n_1 \times n_2}$ with $1 \leq n_1, n_2 \leq T$ and $\|A\|_{\ell^0} = m$ and with $A_{i,j} \in B_K(\{0,1\}^K)$ if $A_{i,j} \neq 0$, then one can store A by storing the values n_1, n_2, the value $0 \leq m \leq T^2$, the position of each of the m non-zero entries of A, and the bit-string of length K that is associated to each non-zero weight (by B_K). Since one can always zero-pad the obtained bit-string to a larger length, and since we have

$$\log_2(T) = \log_2(3d) + \log_2(M) \leq C_1 + \lceil \log_2 M\rceil$$

and $\log_2(1 + T^2) \leq \log_2(2T^2) = 1 + 2\log_2(T) \leq 1 + 2C_1 + 2\lceil \log_2 M\rceil$ for a suitable $C_1 = C_1(d) \in \mathbb{N}$, this can be done with

$$
\begin{aligned}
&\lceil \log_2 T\rceil + \lceil \log_2 T\rceil + \lceil \log_2(T^2+1)\rceil + m \cdot (\lceil \log_2 T\rceil + \lceil \log_2 T\rceil + K) \\
&\leq 2C_1 + 2\lceil \log_2 M\rceil + 1 + 2C_1 + 2\lceil \log_2 M\rceil + m\,(K + 2C_1 + 2\lceil \log_2 M\rceil) \\
&\leq 1 + 4C_1 + 4\lceil \log_2 M\rceil + 2(1 + C_1) \cdot m \cdot (K + \lceil \log_2 M\rceil) \\
&\leq C_2 + 4\lceil \log_2 M\rceil + C_2 \cdot m \cdot (K + \lceil \log_2 M\rceil)
\end{aligned}
$$

bits, for a suitable constant $C_2 = C_2(d) \in \mathbb{N}$.

Likewise, but easier, if $b \in \mathbb{R}^n$ with $1 \leq n \leq T$, with $\|b\|_{\ell^0} = m$ and with $b_i \in B_K(\{0,1\}^K)$ if $b_i \neq 0$, then one can store b by storing the values $1 \leq n \leq T$ and $0 \leq m \leq n \leq T$, and the position of each non-zero entry, as well as the bit-string of length K associated to each non-zero entry (by B_K). Because of $\log_2(T+1) \leq \log_2(2T) \leq 1 + \log_2(T)$, this can be done with

$$\begin{aligned}\lceil \log_2 T \rceil + \lceil \log_2(T+1) \rceil + m \cdot (K + \lceil \log_2 T \rceil) &\leq 1 + 2C_1 + 2\lceil \log_2 M \rceil + m \cdot (K + C_1 + \lceil \log_2 M \rceil) \\ &\leq C_2 + 4\lceil \log_2 M \rceil + C_2 \cdot m \cdot (K + \lceil \log_2 M \rceil)\end{aligned}$$

bits, after possibly enlarging the constant $C_2 = C_2(d) \in \mathbb{N}$ from above.

Note that when decoding a given bit string, the values M, K, d—and thus also of T—are known. Overall, our encoding scheme for encoding networks $\Phi \in \mathcal{NN}^*_{M,K}$ now works as follows:

Step 1: We store the number $1 \leq L \leq M+1$ in a bit-string of length $\lceil \log_2(M+1) \rceil$.

Step 2: We encode each A_ℓ using a bit string of length $C_2 + 4\lceil \log_2 M \rceil + C_2 \cdot \|A_\ell\|_{\ell^0} \cdot (K + \lceil \log_2 M \rceil)$ and each b_ℓ using a bit string of length $C_2 + 4\lceil \log_2 M \rceil + C_2 \cdot \|b_\ell\|_{\ell^0} \cdot (K + \lceil \log_2 M \rceil)$. As seen above, this can indeed be done in such a way that one can uniquely reconstruct $A_1, \dots, A_L$ and $b_1, \dots, b_L$ from these bit-strings, once one knows M, K, d (which are given) and L, which is given by the bit string from Step 1.

Overall, this encodes the network $\Phi = ((A_1, b_1), \dots, (A_L, b_L))$ in a uniquely decodable way using a bit-string of length

$$\begin{aligned}&\lceil \log_2(M+1) \rceil + 2 \cdot \sum_{\ell=1}^{L} (C_2 + 4\lceil \log_2 M \rceil) + C_2 \cdot (K + \lceil \log_2 M \rceil) \sum_{\ell=1}^{L} (\|A_\ell\|_{\ell^0} + \|b_\ell\|_{\ell^0}) \\ &\leq 1 + \lceil \log_2 M \rceil + 2L \cdot (C_2 + 4\lceil \log_2 M \rceil) + C_2 \cdot M \cdot (K + \lceil \log_2 M \rceil) \\ &\leq K + \lceil \log_2 M \rceil + 4 \max\{4, C_2\} \cdot M \cdot (1 + \lceil \log_2 M \rceil) + C_2 \cdot M \cdot (K + \lceil \log_2 M \rceil) \\ &\leq (1 + C_2 + 4 \max\{4, C_2\}) \cdot M \cdot (K + \lceil \log_2 M \rceil).\end{aligned}$$

Here, we used that $L \leq M + 1 \leq 2M$ and that $M, K \geq 1$. With $C := 1 + C_2 + 4 \max\{4, C_2\}$, we have thus proved the claim. ☐

Now, since we have a lower bound on the minimax code-length of the class of horizon functions and since we know how to encode neural networks of limited complexity, we can now prove our optimality result in the uniform setting, by making precise the arguments that we sketched at the beginning of the present subsection.

Proof of Theorem 4.2. We will use the notation $\mathcal{NN}^{\mathcal{B}}_{M,K,d}$ from Definition 4.1 and the notation $\mathcal{NN}^{\mathcal{B},\varrho}_{M,K,d}$ from Lemma B.4. Recall from that lemma that there is an absolute constant $C_1 = C_1(d) \in \mathbb{N}$, such that for arbitrary $M, K \in \mathbb{N}$, there is an injective map

$$\Gamma : \mathcal{NN}^{\mathcal{B},\varrho}_{M,K,d} \to \{0,1\}^{C_1 \cdot M \cdot (K + \lceil \log_2 M \rceil)} .$$

Furthermore, Lemma B.3 yields constants $C_2 = C_2(d, \beta, B) > 0$ and $1/2 > \varepsilon_0 = \varepsilon_0(d, \beta, B) > 0$ such that the minimax code length of $\mathcal{HF}_{\beta,d,B}$ satisfies $L(\varepsilon, \mathcal{HF}_{\beta,d,B}) \geq C_2 \cdot \varepsilon^{-2(d-1)/\beta}$ for all $\varepsilon \in (0, \varepsilon_0)$. Define

$$C := \min \left\{ 1, C_2 \Big/ \left[2C_1 \cdot \left(2 + \frac{2d}{\beta} + C_0 \right) \right] \right\} > 0,$$

fix some $\varepsilon \in (0, \varepsilon_0)$, and define

$$K_0 := \left\lceil C_0 \cdot \log_2 \left(\frac{1}{\varepsilon} \right) \right\rceil \quad \text{and} \quad M_0 := \left\lfloor C \cdot \varepsilon^{-2(d-1)/\beta} \big/ \log_2 \left(\frac{1}{\varepsilon} \right) \right\rfloor .$$

To prove the theorem, it suffices to show that there is $f_\varepsilon \in \mathcal{HF}_{\beta,d,B}$ such that for every $\Phi \in \mathcal{NN}^{\mathcal{B}}_{M,K_0,d}$ (for arbitrary $M \in \mathbb{N}$) with $\|f_\varepsilon - \mathrm{R}_\varrho(\Phi)\|_{L^2} \leq \varepsilon$, it already follows that $M > M_0$.

Assume towards a contradiction that this fails; thus, for *every* $f \in \mathcal{HF}_{\beta,d,B}$, there is $\Phi_f \in \mathcal{NN}^{\mathcal{B}}_{M,K_0,d}$ with $\|f - \mathrm{R}_\varrho(\Phi_f)\|_{L^2} \leq \varepsilon$, but such that $M \leq M_0$. In particular, $\Phi_f \in \mathcal{NN}^{\mathcal{B}}_{M,K_0,d} \subset \mathcal{NN}^{\mathcal{B}}_{M_0,K_0,d}$, so that $\mathrm{R}_\varrho(\Phi_f) \in \mathcal{NN}^{\mathcal{B},\varrho}_{M_0,K_0,d}$.

Let

$$\ell := C_1 \cdot M_0 \cdot (K_0 + \lceil \log_2 M_0 \rceil),$$

and recall from above (or from Lemma B.4) that there is an injection $\Gamma : \mathcal{NN}^{\mathcal{B},\varrho}_{M_0,K_0,d} \to \{0,1\}^\ell$. Therefore, there is a left inverse $\Lambda : \{0,1\}^\ell \to \mathcal{NN}^{\mathcal{B},\varrho}_{M_0,K_0,d}$ for Γ. Using these, we can now define an encoder-decoder pair for $\mathcal{HF}_{\beta,d,B}$, as follows:

$$E^\ell : \mathcal{HF}_{\beta,d,B} \to \{0,1\}^\ell , \qquad f \mapsto \Gamma\left(\mathrm{R}_\varrho(\Phi_f)\right),$$
$$D^\ell : \{0,1\}^\ell \to L^2\left([-1/2,\, 1/2]^d\right),\ c \mapsto [\Lambda(c)]\,|_{[-1/2,1/2]^d}.$$

With this definition, we have

$$D^\ell(E^\ell(f)) = [\Lambda(\Gamma(\mathrm{R}_\varrho(\Phi_f)))]|_{[-1/2,1/2]^d} = \mathrm{R}_\varrho(\Phi_f)|_{[-1/2,1/2]^d},$$

and thus $\|f - D^\ell(E^\ell(f))\|_{L^2} \le \varepsilon$ for all $f \in \mathcal{HF}_{\beta,d,B}$. By definition of the minimax code length $L(\varepsilon, \mathcal{HF}_{\beta,d,B})$, this implies

$$\ell \ge L(\varepsilon, \mathcal{HF}_{\beta,d,B}) \ge C_2 \cdot \varepsilon^{-2(d-1)/\beta}. \tag{B.5}$$

In the remainder of the proof, we use elementary estimates to derive a contradiction to the preceding estimate for ℓ. First, recall $\varepsilon < 1/2$, so that $\log_2(1/\varepsilon) \ge 1$, and hence $M_0 \le C \cdot \varepsilon^{-2(d-1)/\beta} \le \varepsilon^{-2(d-1)/\beta}$, which implies $\log_2 M_0 \le \frac{2(d-1)}{\beta} \cdot \log_2(1/\varepsilon) \le \frac{2d}{\beta} \cdot \log_2(1/\varepsilon)$. Therefore, we get

$$K_0 + \lceil \log_2 M_0 \rceil \le 1 + C_0 \cdot \log_2\left(\frac{1}{\varepsilon}\right) + \left\lceil \frac{2d}{\beta} \cdot \log_2\left(\frac{1}{\varepsilon}\right)\right\rceil \le \left(2 + C_0 + \frac{2d}{\beta}\right) \cdot \log_2\left(\frac{1}{\varepsilon}\right),$$

where the last step used again that $\log_2(1/\varepsilon) \ge 1$. All in all, recalling the definition of M_0 and of C, we see

$$\ell = C_1 \cdot M_0 \cdot (K_0 + \lceil \log_2 M_0 \rceil) \le C_1 \cdot M_0 \cdot \left(2 + \frac{2d}{\beta} + C_0\right) \cdot \log_2\left(\frac{1}{\varepsilon}\right) \le \frac{C_2}{2} \cdot \varepsilon^{-2(d-1)/\beta},$$

which yields the desired contradiction, once we recall Equation (B.5). □

B.2 Lower bounds for the setting of instance optimality

In the previous section, we showed (up to log factors) that $M^{\mathcal{B},\varrho,C_0}_\varepsilon(\mathcal{HF}_{\beta,d,B}) \gtrsim \varepsilon^{-2(d-1)/\beta}$. Here, the quantity $M = M^{\mathcal{B},\varrho,C_0}_\varepsilon(\mathcal{HF}_{\beta,d,B})$ is the minimal $M \in \mathbb{N}$ such that *every* $f \in \mathcal{HF}_{\beta,d,B}$ can be approximated up to an L^2-error of at most ε using a neural network with M non-zero weights (and such that all weights can be encoded with at most $\lceil C_0 \cdot \log_2(1/\varepsilon)\rceil$ bits, using the encoding scheme $\mathcal{B}$).

In this section, we want to show that a similar lower bound holds if one is interested in approximation of a *single* (judiciously chosen) function $f \in \mathcal{HF}_{\beta,d,B}$, not just if one is interested in a uniform approximation over the whole class of horizon functions.

The proof idea is somewhat similar to the one that was used for the lower bounds in the uniform setting: We first obtain a lower bound regarding encoder-decoder pairs which achieve a small L^1-error over the class $\mathcal{F}_{\beta,d-1,B}$, and then we use the map $\gamma \mapsto \mathrm{HF}_\gamma$ to transfer the result to the class of horizon functions.

Thus, our first step is the following lemma which uses Baire's category theorem to "upgrade" the lower bound regarding encoder-decoder pairs with *uniform* error control to a lower bound concerning encoder-decoder pairs with *non-uniform* error control. This is done for the class $\mathcal{F}_{\beta,d-1,B}$.

Lemma B.5. *Let $d \in \mathbb{N}$ and $\beta, B > 0$ be arbitrary, and write $\beta = n + \sigma$ with $n \in \mathbb{N}_0$ and $\sigma \in (0,1]$. Define*

$$X := \{u \in C^n([-1/2, 1/2]^d) \,:\, \|u\|_{C^{0,\beta}} \le B < \infty\}.$$

Let $\phi : \mathbb{N} \to (0,\infty)$ be arbitrary with $\lim_{\ell\to\infty} \ell^{\beta/d} \cdot \phi(\ell) = 0$. Finally, let $I \subset \mathbb{N}$ be infinite, and for each $\ell \in I$, let $E^\ell : X \to \{0,1\}^\ell$ and $D^\ell : \{0,1\}^\ell \to L^1([-1/2, 1/2]^d)$ be arbitrary maps.

Then there is some $u \in X$, such that the sequence $(\|u - D^\ell(E^\ell(u))\|_{L^1} \,/\, \phi(\ell))_{\ell\in I}$ is unbounded.

Proof. We assume towards a contradiction that the claim is false. This means

$$\forall\, u \in X : \left(\|u - D^\ell(E^\ell(u))\|_{L^1} \,/\, \phi(\ell)\right)_{\ell\in I} \text{ is a bounded sequence.} \tag{B.6}$$

In the following, we consider the Banach space

$$C^{0,\beta}([-1/2, 1/2]^d) := \{u \in C^n([-1/2, 1/2]^d) \,:\, \|u\|_{C^{0,\beta}} < \infty\},$$

i.e., all balls $B_\delta(u)$ or $\overline{B}_\delta(u)$ for $u \in C^{0,\beta}$, and all closures $\overline{M}$ for $M \subset C^{0,\beta}$ are to be understood with respect to the $\|\cdot\|_{C^{0,\beta}}$ norm.

We divide the proof into three steps.

Step 1: For $N \in \mathbb{N}$, let us set

$$G_N := \left\{ u \in X \,:\, \forall \ell \in I : \|u - D^\ell(E^\ell(u))\|_{L^1} \leq N \cdot \phi(\ell) \right\} .$$

In this step, we show that there is some $N \in \mathbb{N}$ and certain $\delta > 0$ and $u_0 \in X$ with

$$\overline{B_\delta}(u_0) \subset \overline{G_N}. \tag{B.7}$$

To see this, first note that Equation (B.6) simply says $X = \bigcup_{N\in\mathbb{N}} G_N$. But X is a closed subspace of the Banach space $C^{0,\beta}([-1/2, 1/2]^d)$, and thus a complete metric space. Therefore, the Baire category theorem (see e.g. [19, Theorem 5.9]) shows that at least one of the G_N has nonempty interior (with respect to X). In other words, Baire's theorem ensures existence of some $N \in \mathbb{N}$ and of $\delta_0 \in (0, 1)$ and $v_0 \in X$ such that

$$X \cap B_{\delta_0}(v_0) \subset \overline{G_N},$$

where the ball $B_{\delta_0}(v_0)$ and the closure $\overline{G_N}$ are both formed with respect to the norm $\|\cdot\|_{C^{0,\beta}}$.

Now, set $u_0 := (1 - \delta_0/(1 + B)) \cdot v_0$ and note

$$\|u_0\|_{C^{0,\beta}} = (1 - \delta_0/(1 + B)) \cdot \|v_0\|_{C^{0,\beta}} \leq (1 - \delta_0/(1 + B)) \cdot B < B,$$

as well as $\|u_0 - v_0\|_{C^{0,\beta}} = \frac{\delta_0}{1+B} \cdot \|v_0\|_{C^{0,\beta}} < \delta_0$. These two properties easily imply that there is some $\delta > 0$ with $\overline{B_\delta}(u_0) \subset X \cap B_{\delta_0}(v_0)$. Because of $X \cap B_{\delta_0}(v_0) \subset \overline{G_N}$, this establishes Equation (B.7).

Step 2: For brevity, set

$$Y := \overline{B_\delta}(0) = \{u \in C^{0,\beta}([-1/2, 1/2]^d) \,:\, \|u\|_{C^{0,\beta}} \leq \delta\} = \mathcal{F}_{\beta,d,\delta},$$

where the notation $\mathcal{F}_{\beta,d,\delta}$ is as in Equation (3.1). Our goal in this step is for each $\ell \in I \subset \mathbb{N}$ to construct modified maps $\widetilde{D^\ell} : \{0, 1\}^\ell \to L^1([-1/2, 1/2]^d)$ and $\widetilde{E^\ell} : Y \to \{0, 1\}^\ell$ which satisfy

$$\|u - \widetilde{D^\ell}(\widetilde{E^\ell}(u))\|_{L^1} \leq N \cdot \phi(\ell) \qquad \text{for all } u \in Y \text{ and all } \ell \in I. \tag{B.8}$$

To this end, define

$$\widetilde{D^\ell} : \{0, 1\}^\ell \to L^1([-1/2, 1/2]^d),\ c \mapsto D^\ell(c) - u_0.$$

Now, since $\{0, 1\}^\ell$ is finite, there is for each $u \in Y$ a certain (not necessarily unique) coefficient sequence $c_u \in \{0, 1\}^\ell$ with

$$\|u - \widetilde{D^\ell}(c_u)\|_{L^1} = \min_{c\in\{0,1\}^\ell} \|u - \widetilde{D^\ell}(c)\|_{L^1} .$$

With this choice of c_u, we define $\widetilde{E^\ell} : Y \to \{0, 1\}^\ell, u \mapsto c_u$. To prove Equation (B.8) recall for $u \in Y$ from Step 1 that $u + u_0 \in \overline{B_\delta}(u_0) \subset \overline{G_N}$. Thus, there is a sequence $(u_k)_{k\in\mathbb{N}}$ in G_N with $\|(u + u_0) - u_k\|_{C^{0,\beta}} \to 0$ as $k \to \infty$. In particular, we get

$$\begin{aligned}
\|u - \widetilde{D^\ell}(\widetilde{E^\ell}(u))\|_{L^1} &= \min_{c\in\{0,1\}^\ell} \|u - \widetilde{D^\ell}(c)\|_{L^1} \leq \|u - \widetilde{D^\ell}(E^\ell(u_k))\|_{L^1} \\
&= \|(u + u_0) - D^\ell(E^\ell(u_k))\|_{L^1} \\
&\leq \|(u + u_0) - u_k\|_{L^1} + \|u_k - D^\ell(E^\ell(u_k))\|_{L^1} \\
\left(\text{since } u_k\in G_N \text{ and } \|\cdot\|_{L^1([-1/2,1/2]^d)}\leq\|\cdot\|_{C^{0,\beta}}\right) &\leq \|(u + u_0) - u_k\|_{C^{0,\beta}} + N \cdot \phi(\ell) \xrightarrow[k\to\infty]{} N \cdot \phi(\ell),
\end{aligned}$$

which is precisely what was claimed in (B.8).

Step 3: In this step, we complete the proof. To this end, recall from Step 1 of the proof of Lemma B.3 that there are constants $C = C(\beta, d, \delta) > 0$ and $\varepsilon_0 = \varepsilon_0(\beta, d, \delta) > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$, there is some $N \geq \exp(C \cdot \varepsilon^{-d/\beta})$ and certain functions $u_1, \dots, u_N \in Y = \mathcal{F}_{\beta,d,\delta}$ satisfying $\|u_i - u_j\|_{L^1} \geq \varepsilon$ for $i \neq j$.

We now apply this for every fixed, sufficiently large $\ell \in I$ with the choice $\varepsilon = (C^{-1} \cdot \ell)^{-\beta/d}$. Note that we indeed have $\varepsilon \in (0, \varepsilon_0)$, once ℓ is large enough, which we always assume in the following; since $I \subset \mathbb{N}$ is infinite, there exist arbitrarily large $\ell \in I$. As just seen, there is some $N \geq \exp(C \cdot [(C^{-1} \cdot \ell)^{-\beta/d}]^{-d/\beta}) = e^\ell$, and certain functions $u_1, \dots, u_N \in Y$ with $\|u_i - u_j\|_{L^1} \geq \varepsilon = (C^{-1} \cdot \ell)^{-\beta/d}$ for $i \neq j$.

Because of $N \geq e^{\ell} > 2^{\ell} = |\{0,1\}^{\ell}|$, the pigeonhole principle shows that there are $i, j \in \{1, \dots, N\}$ with $i \neq j$, but such that $\widetilde{E^{\ell}}(u_i) = \widetilde{E^{\ell}}(u_j)$. In view of Equation (B.8), this implies

$$
\begin{aligned}
(C^{-1} \cdot \ell)^{-\beta/d} &\leq \|u_i - u_j\|_{L^1} \\
&\leq \|u_i - \widetilde{D^{\ell}}(\widetilde{E^{\ell}}(u_i))\|_{L^1} + \|\widetilde{D^{\ell}}(\widetilde{E^{\ell}}(u_i)) - \widetilde{D^{\ell}}(\widetilde{E^{\ell}}(u_j))\|_{L^1} + \|\widetilde{D^{\ell}}(\widetilde{E^{\ell}}(u_j)) - u_j\|_{L^1} \\
&\leq N \cdot \phi(\ell) + 0 + N \cdot \phi(\ell).
\end{aligned}
$$

By rearranging, and by our assumption on ϕ, this implies

$$
\frac{C^{\beta/d}}{2N} \leq \ell^{\beta/d} \cdot \phi(\ell) \xrightarrow[\ell \in I, \ell \to \infty]{} 0,
$$

which is the desired contradiction, since the left-hand side is positive and independent of ℓ. Note that we again used that I is infinite to ensure that the limit $\ell \in I, \ell \to \infty$ makes sense. ☐

Our next result transfers the preceding lemma from the set $\mathcal{F}_{\beta,d-1,B}$ of smooth functions to the class $\mathcal{HF}_{\beta,d,B}$ of horizon functions.

Lemma B.6. *Let $d \in \mathbb{N}_{\geq 2}$ and $\beta, B > 0$ be arbitrary. Furthermore, let $\vartheta : \mathbb{N} \to (0, \infty)$ be arbitrary with $\lim_{\ell \to \infty} \ell^{\beta/(2(d-1))} \cdot \vartheta(\ell) = 0$. Finally, let $I \subset \mathbb{N}$ be infinite, and for each $\ell \in I$ let $E^{\ell} : \mathcal{HF}_{\beta,d,B} \to \{0,1\}^{\ell}$ and $D^{\ell} : \{0,1\}^{\ell} \to L^2([-1/2, 1/2]^d)$ be arbitrary.*

Then there is some $f \in \mathcal{HF}_{\beta,d,B}$ such that the sequence

$$
\big(\|f - D^{\ell}(E^{\ell}(f))\|_{L^2} \,/\, \vartheta(\ell)\big)_{\ell \in I}
$$

is unbounded.

Proof. Write $\beta = n + \sigma$ with $n \in \mathbb{N}_0$ and $\sigma \in (0, 1]$.

Step 1: We show for arbitrary $C > 0$ that the set

$$
K_C := \{f \in C^n([-1/2, 1/2]^{d-1}) \, : \, \|f\|_{C^{0,\beta}} \leq C\}
$$

is a compact subset of $L^1([-1/2, 1/2]^{d-1})$. To see this, let $(f_k)_{k \in \mathbb{N}}$ be an arbitrary sequence in K_C. Then, for each $\alpha \in \mathbb{N}_0^{d-1}$ with $|\alpha| < n$, we have

$$
\mathrm{Lip}_1(\partial^{\alpha} f_k) \leq \|\nabla(\partial^{\alpha} f_k)\|_{L^{\infty}} \leq \sum_{j=1}^{d-1} \|\partial^{\alpha + e_j} f_k\|_{L^{\infty}} \leq (d-1) \cdot \|f_k\|_{C^{0,\beta}} \leq d \cdot C,
$$

and for $\alpha \in \mathbb{N}_0^{d-1}$ with $|\alpha| = n$, we have $\mathrm{Lip}_{\sigma}(\partial^{\alpha} f_k) \leq \|f_k\|_{C^{0,\beta}} \leq C$, where we emphasize that $\sigma > 0$. Furthermore, for $|\alpha| \leq n$ arbitrary, we have $\|\partial^{\alpha} f_k\|_{L^{\infty}} \leq \|f_k\|_{C^{0,\beta}} \leq C$.

We have thus shown that each of the sequences $(\partial^{\alpha} f_k)_{k \in \mathbb{N}}$, for $|\alpha| \leq n$, is uniformly bounded and equicontinuous. By the Arzela-Ascoli theorem (see e.g. [19, Theorem 4.44]), there is thus a common subsequence $(f_{k_t})_{t \in \mathbb{N}}$ such that $(\partial^{\alpha} f_{k_t})_{t \in \mathbb{N}}$ converges uniformly to a continuous function $g_{\alpha} \in C([-1/2, 1/2]^{d-1})$ for each $\alpha \in \mathbb{N}_0^{d-1}$ with $|\alpha| \leq n$.

It is now a standard result (see for example [30, Theorem 9.1 in XIII, §9]) that $f := g_0$ satisfies $f \in C^n([-1/2, 1/2]^{d-1})$ with $\partial^{\alpha} f = g_{\alpha}$ for $\alpha \in \mathbb{N}_0^{d-1}$ with $|\alpha| \leq n$. In particular, $f_{k_t} \to g_0 = f$ uniformly, and thus also in $L^1([-1/2, 1/2]^{d-1})$. Thus, to prove compactness of $K_C \subset L^1([-1/2, 1/2]^{d-1})$, it suffices to show $f \in K_C$. But for $\alpha \in \mathbb{N}_0^{d-1}$ with $|\alpha| \leq n$, we have $\|\partial^{\alpha} f\|_{L^{\infty}} = \|g_{\alpha}\|_{L^{\infty}} = \lim_{t \to \infty} \|\partial^{\alpha} f_{k_t}\|_{L^{\infty}} \leq C$. Finally, for $|\alpha| = n$, and arbitrary $x, y \in [-1/2, 1/2]^{d-1}$, we have

$$
\begin{aligned}
|\partial^{\alpha} f(x) - \partial^{\alpha} f(y)| = |g_{\alpha}(x) - g_{\alpha}(y)| &= \lim_{t \to \infty} |\partial^{\alpha} f_{k_t}(x) - \partial^{\alpha} f_{k_t}(y)| \\
&\leq \limsup_{t \to \infty} \mathrm{Lip}_{\sigma}(\partial^{\alpha} f_{k_t}) \cdot |x - y|^{\sigma} \leq \sup_{k \in \mathbb{N}} \|f_k\|_{C^{0,\beta}} \cdot |x - y|^{\sigma} \leq C \cdot |x - y|^{\sigma}.
\end{aligned}
$$

Therefore, $\mathrm{Lip}_{\sigma}(\partial^{\alpha} f) \leq C < \infty$. All in all, we have thus verified $\|f\|_{C^{0,\beta}} \leq C$, i.e, $f \in K_C$.

Step 2: We observe with Lemma B.1 that

$$
\Lambda : L^1([-1/2, 1/2]^{d-1}) \to L^2([-1/2, 1/2]^d), \gamma \mapsto \mathrm{HF}_{\gamma}
$$

with HF_{γ} as in Lemma B.1 is continuous.

Step 3: Let $B_0 := \min\{B, 1/2\}$. In this step, we construct modified encoding-decoding pairs $(\widetilde{E^\ell}, \widetilde{D^\ell})$ with $\widetilde{E^\ell} : \mathcal{F}_{\beta,d-1,B_0} \to \{0,1\}^\ell$ and $\widetilde{D^\ell} : \{0,1\}^\ell \to L^1([-1/2, 1/2]^{d-1})$ such that

$$\|\gamma - \widetilde{D^\ell}(\widetilde{E^\ell}(\gamma))\|_{L^1([-1/2,1/2]^{d-1})} \leq 4 \cdot \|\mathrm{HF}_\gamma - D^\ell(E^\ell(\mathrm{HF}_\gamma))\|^2_{L^2([-1/2,1/2]^d)} \qquad \text{for all } \gamma \in \mathcal{F}_{\beta,d-1,B_0}. \quad \text{(B.9)}$$

For the construction, first note from Steps 1 and 2 that there is for each $g \in L^2([-1/2, 1/2]^d)$ some (not necessarily unique) $\gamma_g \in K_{B_0}$ with $\|g - \mathrm{HF}_{\gamma_g}\|_{L^2} = \min_{\gamma \in K_{B_0}} \|g - \mathrm{HF}_\gamma\|_{L^2}$. Now, for each $c \in \{0,1\}^\ell$, let $\theta_c := \gamma_{D^\ell(c)} \in K_{B_0} \subset L^1([-1/2, 1/2]^{d-1})$, so that

$$\|D^\ell(c) - \mathrm{HF}_{\theta_c}\|_{L^2} = \min_{\gamma \in K_{B_0}} \|D^\ell(c) - \mathrm{HF}_\gamma\|_{L^2} \qquad \text{for all } c \in \{0,1\}^\ell.$$

With this choice, let

$$\widetilde{D^\ell} : \{0,1\}^\ell \to L^1([-1/2, 1/2]^{d-1}),\ c \mapsto \theta_c .$$

Now, since $\{0,1\}^\ell$ is finite, there is for each $\gamma \in \mathcal{F}_{\beta,d-1,B_0}$ some (not necessarily unique) $c_\gamma \in \{0,1\}^\ell$ with $\|\gamma - \widetilde{D^\ell}(c_\gamma)\|_{L^1} = \min_{c \in \{0,1\}^\ell} \|\gamma - \widetilde{D^\ell}(c)\|_{L^1}$. With this choice, set

$$\widetilde{E^\ell} : \mathcal{F}_{\beta,d-1,B_0} \to \{0,1\}^\ell,\ \gamma \mapsto c_\gamma.$$

Now that we have constructed $\widetilde{E^\ell}, \widetilde{D^\ell}$, it remains to establish Equation (B.9). To this end, recall from Lemma B.1 that all $\gamma, \psi \in L^1([-1/2, 1/2]^{d-1})$ with $\|\gamma\|_{\sup}, \|\psi\|_{\sup} \leq 1/2$ satisfy $\|\mathrm{HF}_\gamma - \mathrm{HF}_\psi\|^2_{L^2} = \|\gamma - \psi\|_{L^1}$. Therefore, we get for arbitrary $\gamma \in \mathcal{F}_{\beta,d-1,B_0}$ with $c^{(0)} := E^\ell(\mathrm{HF}_\gamma)$ that

$$\begin{aligned}
\|\gamma - \widetilde{D^\ell}(\widetilde{E^\ell}(\gamma))\|_{L^1} &= \min_{c \in \{0,1\}^\ell} \|\gamma - \widetilde{D^\ell}(c)\|_{L^1} \leq \|\gamma - \widetilde{D^\ell}(c^{(0)})\|_{L^1} = \|\gamma - \theta_{c^{(0)}}\|_{L^1} \\
\left(\substack{\|\gamma\|_{\sup}, \|\theta_{c^{(0)}}\|_{\sup} \leq B_0 \leq \frac{1}{2} \\ \text{since } \gamma \in \mathcal{F}_{\beta,d-1,B_0} \text{ and } \theta_{c^{(0)}} \in K_{B_0}}\right) &= \|\mathrm{HF}_\gamma - \mathrm{HF}_{\theta_{c^{(0)}}}\|^2_{L^2} \\
&\leq \Big(\|\mathrm{HF}_\gamma - D^\ell(c^{(0)})\|_{L^2} + \|D^\ell(c^{(0)}) - \mathrm{HF}_{\theta_{c^{(0)}}}\|_{L^2}\Big)^2 \\
(\text{choice of } \theta_{c^{(0)}}) &= \Big(\|\mathrm{HF}_\gamma - D^\ell(c^{(0)})\|_{L^2} + \min_{\psi \in K_{B_0}} \|D^\ell(c^{(0)}) - \mathrm{HF}_\psi\|_{L^2}\Big)^2 \\
(\text{since } \gamma \in \mathcal{F}_{\beta,d-1,B_0} = K_{B_0}) &\leq \Big(2 \cdot \|\mathrm{HF}_\gamma - D^\ell(c^{(0)})\|_{L^2}\Big)^2 = 4 \cdot \|\mathrm{HF}_\gamma - D^\ell(E^\ell(\mathrm{HF}_\gamma))\|^2_{L^2} .
\end{aligned}$$

This completes the proof of Equation (B.9).

Step 4: In this step, we complete the proof. To this end, let us assume towards a contradiction that the claim fails. Thus, for *every* $f \in \mathcal{HF}_{\beta,d,B}$, we have

$$\|f - D^\ell(E^\ell(f))\|_{L^2} \leq C_f \cdot \vartheta(\ell) \qquad \text{for all } \ell \in I,$$

for a finite constant $C_f > 0$.

By Step 3, this implies for $\phi := \vartheta^2$ and arbitrary $\gamma \in \mathcal{F}_{\beta,d-1,B_0}$ because of $\mathrm{HF}_\gamma \in \mathcal{HF}_{\beta,d,B_0} \subset \mathcal{HF}_{\beta,d,B}$ that

$$\|\gamma - \widetilde{D^\ell}(\widetilde{E^\ell}(\gamma))\|_{L^1} \leq 4 \cdot \|\mathrm{HF}_\gamma - D^\ell(E^\ell(\mathrm{HF}_\gamma))\|^2_{L^2} \leq 4 \cdot C^2_{\mathrm{HF}_\gamma} \cdot (\vartheta(\ell))^2 = 4 \cdot C^2_{\mathrm{HF}_\gamma} \cdot \phi(\ell) \qquad \text{for all } \ell \in I.$$

But by assumption on ϑ, we have $\lim_{\ell \to \infty} \ell^{\beta/(d-1)} \cdot \phi(\ell) = \lim_{\ell \to \infty} (\ell^{\beta/[2(d-1)]} \cdot \vartheta(\ell))^2 = 0$, so that Lemma B.5 yields the desired contradiction. □

With the preceding lemma, we have shown that given a sequence of encoder-decoder pairs for the class of horizon functions, one can always find a *single function* which is not "too well approximated" by the sequence. We now use this result to prove the claimed lower bound in the setting of instance optimality.

Proof of Theorem 4.3. **Step 1**: For technical reasons, we first need to study for fixed, but arbitrary $\nu > 0$ the monotonicity of the function

$$\phi : (2, \infty) \to (0, \infty), x \mapsto \frac{x^\nu}{\log_2(x) \cdot \log_2(\log_2(x))}.$$

We claim that there is some $x_0 = x_0(\nu)$ such that $\phi|_{[x_0,\infty)}$ is strictly increasing; since we clearly have $\phi(x) \to \infty$ as $x \to \infty$, we can then choose x_0 so that also $\phi(x_0) \geq 4$.

To show existence of x_0, first note from a direct computation that

$$\phi'(x) = x^{\nu-1} \cdot \left[\nu \cdot \log_2(x) \cdot \log_2(\log_2(x)) - \frac{\log_2(\log_2(x))}{\ln 2} - (\ln 2)^{-2}\right] \Big/ [\log_2(x) \cdot \log_2(\log_2(x))]^2.$$

Here, the denominator is positive. Furthermore, the first term in the numerator dominates the other two terms for x large enough. Therefore, $\phi'(x)$ is positive for x large enough. This establishes the claim of Step 1.

Step 2: In this technical step, we construct quantities $\Omega_\varepsilon, K_\varepsilon, \ell_\varepsilon \in \mathbb{N}$ for $0 < \varepsilon \le \varepsilon_0$, for a certain $\varepsilon_0 \in (0, 1/4]$, and use these quantities to define an infinite set $I \subset \mathbb{N}$. The relevance of these constructions will become apparent in Steps 3 and 4.

Let ϕ, x_0 be as in Step 1, with $\nu := 2(d-1)/\beta$. By possibly enlarging x_0, we can (and will) assume $x_0 \ge 4$. Set $\varepsilon_0 := x_0^{-1}$, choose the constant $C = C(d) \in \mathbb{N}$ as provided by Lemma B.4, and let $C_1 = C_1(d, \beta, C_0) \in \mathbb{N}$ with $C_1 \ge C_0^{-1} \cdot (1 + 2(d-1)/\beta)$. Furthermore, set $C_2 := C \cdot (1 + C_1) \in \mathbb{N}$.

Next, for $\varepsilon \in (0, \varepsilon_0]$, define

$$\Omega_\varepsilon := \lceil \phi(\varepsilon^{-1}) \rceil = \left\lceil \varepsilon^{-2(d-1)/\beta} \big/ \left[\log_2\left(\varepsilon^{-1}\right) \cdot \log_2\left(\log_2\left(\varepsilon^{-1}\right)\right)\right] \right\rceil \in \mathbb{N}$$

and $K_\varepsilon := \lceil C_0 \cdot \log_2(1/\varepsilon) \rceil \in \mathbb{N}$, and set $\ell_\varepsilon := C_2 \cdot \Omega_\varepsilon \cdot K_\varepsilon \in \mathbb{N}$.

First, note because of $0 < \varepsilon \le \varepsilon_0 \le 1/4$ that $\varepsilon^{-1} \ge 4$ and hence $\log_2(1/\varepsilon) \ge 2$ and $\log_2(\log_2(1/\varepsilon)) \ge 1$, as well as $\varepsilon^{-2(d-1)/\beta} \ge 1$, Hence, $\Omega_\varepsilon \le \lceil \varepsilon^{-2(d-1)/\beta} \rceil \le 1 + \varepsilon^{-2(d-1)/\beta} \le 2 \cdot \varepsilon^{-2(d-1)/\beta}$, which implies

$$\log_2(\Omega_\varepsilon) \le 1 + \frac{2(d-1)}{\beta} \cdot \log_2\left(\frac{1}{\varepsilon}\right) \le \left(1 + \frac{2(d-1)}{\beta}\right) \cdot \log_2\left(\frac{1}{\varepsilon}\right) \le C_1 \cdot C_0 \cdot \log_2\left(\frac{1}{\varepsilon}\right) \le C_1 \cdot K_\varepsilon.$$

Therefore,

$$C \cdot \Omega_\varepsilon \cdot (K_\varepsilon + \lceil \log_2 \Omega_\varepsilon \rceil) \le C \cdot (1 + C_1) \cdot \Omega_\varepsilon \cdot K_\varepsilon = \ell_\varepsilon. \tag{B.10}$$

Our last goal in this step is to show that the map $\ell_\varepsilon \mapsto (\Omega_\varepsilon, K_\varepsilon)$ is well-defined. To see this, first recall from Step 1, that if $0 < \varepsilon \le \varepsilon'$, then $\Omega_\varepsilon \ge \Omega_{\varepsilon'}$. By contraposition, this shows that if $\Omega_\varepsilon < \Omega_{\varepsilon'}$, then $\varepsilon > \varepsilon'$ and hence $K_\varepsilon \le K_{\varepsilon'}$, so that

$$\ell_\varepsilon = C_2 \cdot \Omega_\varepsilon \cdot K_\varepsilon \le C_2 \cdot \Omega_\varepsilon \cdot K_{\varepsilon'} < C_2 \cdot \Omega_{\varepsilon'} \cdot K_{\varepsilon'} = \ell_{\varepsilon'}.$$

Again by contraposition, we have shown that $\Omega_\varepsilon = \Omega_{\varepsilon'}$ if $\ell_\varepsilon = \ell_{\varepsilon'}$. Even more, if $\ell_\varepsilon = \ell_{\varepsilon'}$, we just saw $\Omega_\varepsilon = \Omega_{\varepsilon'}$, but this also implies $K_\varepsilon = \ell_\varepsilon/(C_2\Omega_\varepsilon) = \ell_{\varepsilon'}/(C_2\Omega_{\varepsilon'}) = K_{\varepsilon'}$. Hence, if we define $I := \{\ell_\varepsilon : \varepsilon \in (0, x_0]\} \subset \mathbb{N}$, then I is clearly an infinite set, and for $\ell = \ell_\varepsilon \in I$, it makes sense to write $\Omega_\varepsilon, K_\varepsilon$, since these quantities are independent of the precise choice of $\varepsilon \in (0, x_0]$ with $\ell = \ell_\varepsilon$.

Step 3: In this step, we define for each $\ell \in I$ a certain encoder-decoder pair (E^ℓ, D^ℓ). More precisely, we recall from Lemma B.4 by our choice of $C = C(d)$ in Step 2 and because of Equation (B.10) that for each $\ell = \ell_\varepsilon \in I$, there is an injective function $\Gamma_\ell : \mathcal{NN}^{\mathcal{B},\varrho}_{\Omega_\varepsilon, K_\varepsilon, d} \to \{0,1\}^\ell$. Let us fix some left-inverse $\Psi_\ell : \{0,1\}^\ell \to \mathcal{NN}^{\mathcal{B},\varrho}_{\Omega_\varepsilon, K_\varepsilon, d}$ for Γ_ℓ.

Next, for each $\ell = \ell_\varepsilon \in I$ and each $f \in \mathcal{HF}_{\beta,d,B}$ we can use finiteness of $\mathcal{NN}^{\mathcal{B},\varrho}_{\Omega_\varepsilon, K_\varepsilon, d}$ (which follows from the injectivity of Γ_ℓ) to choose a (not necessarily unique) neural network $\Phi_{f,\ell} \in \mathcal{NN}^{\mathcal{B}}_{\Omega_\varepsilon, K_\varepsilon, d}$ which satisfies

$$\|f - \mathrm{R}_\varrho(\Phi_{f,\ell})\|_{L^2} = \min_{N \in \mathcal{NN}^{\mathcal{B}}_{\Omega_\varepsilon, K_\varepsilon, d}} \|f - \mathrm{R}_\varrho(\Phi)\|_{L^2}.$$

With this choice, we can finally define

$$\begin{aligned} E^\ell &: \mathcal{HF}_{\beta,d,B} \to \{0,1\}^\ell, \qquad f \mapsto \Gamma_\ell\left(\mathrm{R}_\varrho(\Phi_{f,\ell})\right), \\ D^\ell &: \{0,1\}^\ell \to L^2\left([-1/2, 1/2]^d\right), c \mapsto [\Psi_\ell(c)]\,|_{[-\frac{1}{2},\frac{1}{2}]^d}. \end{aligned}$$

Note that this choice implies

$$\|f - D^\ell(E^\ell(f))\|_{L^2} = \min_{\Phi \in \mathcal{NN}^{\mathcal{B}}_{\Omega_\varepsilon, K_\varepsilon, d}} \|f - \mathrm{R}_\varrho(\Phi)\|_{L^2} \qquad \text{for all } \ell = \ell_\varepsilon \in I \text{ and } f \in \mathcal{HF}_{\beta,d,B}. \tag{B.11}$$

Step 4: In Step 5, we will invoke Lemma B.6 with

$$\vartheta : \mathbb{N} \to (0, \infty), \ell \mapsto \ell^{-\beta/(2(d-1))} \big/ [\log_2(\log_2(\max\{4, \ell\}))]^{\beta/(2(d-1))}.$$

As a preparation, in this step, we derive some elementary estimates concerning $\Omega_\varepsilon, K_\varepsilon$ and ℓ_ε, and then also for $\vartheta(\ell_\varepsilon)$.

First, note for $\varepsilon \in (0, \varepsilon_0]$ because of $\varepsilon_0 \leq 1/4$ that $\log_2(1/\varepsilon) \geq 2 \geq 1$, and hence

$$K_\varepsilon = \left\lceil C_0 \cdot \log_2\left(\frac{1}{\varepsilon}\right)\right\rceil \leq 1 + C_0 \log_2\left(\frac{1}{\varepsilon}\right) \leq (1 + C_0) \cdot \log_2\left(\frac{1}{\varepsilon}\right).$$

Next, since $\phi(\varepsilon^{-1}) \geq \phi(\varepsilon_0^{-1}) = \phi(x_0) \geq 4$ for $\varepsilon \in (0, \varepsilon_0]$, we have $\Omega_\varepsilon = \lceil \phi(\varepsilon^{-1}) \rceil \leq 1 + \phi(\varepsilon^{-1}) \leq 2\phi(\varepsilon^{-1})$. All in all, this yields for a suitable constant $C_3 = C_3(d, \beta, C_0) \in \mathbb{N}$ that

$$\ell_\varepsilon = C_2 \cdot \Omega_\varepsilon \cdot K_\varepsilon \leq 2 \cdot (1 + C_0) \cdot C_2 \cdot \phi\left(\frac{1}{\varepsilon}\right) \cdot \log_2\left(\frac{1}{\varepsilon}\right) = C_3 \cdot \varepsilon^{-2(d-1)/\beta} \cdot \left[\log_2\left(\log_2\left(\frac{1}{\varepsilon}\right)\right)\right]^{-1}.$$

Furthermore, because of $\log_2(\log_2(1/\varepsilon)) \geq 1$, we get for a suitable constant $C_4 = C_4(d, \beta, C_0) \in \mathbb{N}$ that

$$\log_2(\ell_\varepsilon) \leq \log_2(C_3 \cdot \varepsilon^{-2(d-1)/\beta}) = \log_2(C_3) + \frac{2(d-1)}{\beta} \cdot \log_2\left(\frac{1}{\varepsilon}\right) \leq C_4 \cdot \log_2\left(\frac{1}{\varepsilon}\right),$$

so that $\log_2(\log_2(\ell_\varepsilon)) \leq \log_2(C_4) + \log_2(\log_2(1/\varepsilon)) \leq C_5 \cdot \log_2(\log_2(1/\varepsilon))$ for some $C_5 = C_5(d, \beta, C_0) > 0$.

From the preceding estimates, because of $\ell_\varepsilon = C_2 \cdot \Omega_\varepsilon \cdot K_\varepsilon \geq \Omega_\varepsilon \geq \phi(\varepsilon^{-1}) \geq 4$, and from the definition of ϑ, we get a constant $C_6 = C_6(d, \beta, C_0) > 0$ with

$$\begin{aligned} \frac{1}{\vartheta(\ell_\varepsilon)} &= \ell_\varepsilon^{\beta/(2(d-1))} \cdot [\log_2(\log_2(\ell_\varepsilon))]^{\beta/(2(d-1))} \\ &\leq C_3^{\beta/(2(d-1))} \cdot \varepsilon^{-1} \cdot [\log_2(\log_2(1/\varepsilon))]^{-\beta/(2(d-1))} \cdot C_5^{\beta/(2(d-1))} \cdot [\log_2(\log_2(1/\varepsilon))]^{\beta/(2(d-1))} \\ &= C_6 \cdot \varepsilon^{-1}. \end{aligned} \tag{B.12}$$

Step 5: Now, we complete the proof. First, we note

$$\lim_{\ell \to \infty} \ell^{\beta/(2(d-1))} \vartheta(\ell) = \lim_{\ell \to \infty} [\log_2(\log_2(\max\{4, \ell\}))]^{-\beta/(2(d-1))} = 0,$$

as required in Lemma B.6. Hence, using that lemma, we obtain a horizon function $f \in \mathcal{HF}_{\beta,d,B}$ which satisfies

$$\begin{aligned} \infty &= \sup_{\ell_\varepsilon \in I} \frac{\|f - D^{\ell_\varepsilon}(E^{\ell_\varepsilon}(f))\|_{L^2}}{\vartheta(\ell_\varepsilon)} \\ (\text{by Equations (B.11) and (B.12)}) &\leq C_6 \cdot \sup_{0 < \varepsilon \leq \varepsilon_0} \left[\min\{\|f - \mathrm{R}_\varrho(\Phi)\|_{L^2} \,:\, \Phi \in \mathcal{NN}^{\mathcal{B}}_{\Omega_\varepsilon, K_\varepsilon, d}\} \cdot \varepsilon^{-1}\right]. \end{aligned}$$

For brevity, let us set $\delta_\varepsilon := \min\{\|f - \mathrm{R}_\varrho(\Phi)\|_{L^2} \,:\, \Phi \in \mathcal{NN}^{\mathcal{B}}_{\Omega_\varepsilon, K_\varepsilon, d}\}$. Then the preceding estimate yields a sequence $(\varepsilon_k)_{k \in \mathbb{N}}$ with $0 < \varepsilon_k \leq \varepsilon_0 \leq 1/4$ and such that $\varepsilon_k^{-1} \cdot \delta_{\varepsilon_k} \geq 2k$ for all $k \in \mathbb{N}$. In particular, $\delta_{\varepsilon_k} > 0$.

But for $0 < \varepsilon \leq \varepsilon' \leq \varepsilon_0$, we have $\Omega_\varepsilon \geq \Omega_{\varepsilon'}$ and $K_\varepsilon \geq K_{\varepsilon'}$, see also Step 2. Therefore, and since we require the encoding scheme $\mathcal{B} = (B_\ell)_{\ell \in \mathbb{N}}$ to be consistent, i.e., to satisfy $\mathrm{Range}(B_\ell) \subset \mathrm{Range}(B_{\ell+1})$ for all $\ell \in \mathbb{N}$, we have $\mathcal{NN}^{\mathcal{B}}_{\Omega_\varepsilon, K_\varepsilon, d} \supset \mathcal{NN}^{\mathcal{B}}_{\Omega_{\varepsilon'}, K_{\varepsilon'}, d}$, and thus $\delta_\varepsilon \leq \delta_{\varepsilon'}$. In particular, we get

$$0 < \varepsilon_k \leq \frac{\delta_{\varepsilon_k}}{2k} \leq \frac{\delta_{\varepsilon_0}}{2k} \xrightarrow[k \to \infty]{} 0.$$

We have thus constructed the function $f \in \mathcal{HF}_{\beta,d,B}$ and the sequence $(\varepsilon_k)_{k \in \mathbb{N}}$, so that it remains to show that these have the desired properties. To see this, pick any $k \in \mathbb{N}$, and let $M \in \mathbb{N}$ such that there exists $\Phi \in \mathcal{NN}^{\mathcal{B}}_{M, \lceil C_0 \log_2(1/\varepsilon_k) \rceil, d} = \mathcal{NN}^{\mathcal{B}}_{M, K_{\varepsilon_k}, d}$ with $\|f - \mathrm{R}_\varrho(\Phi)\|_{L^2} \leq \varepsilon_k$. Then we get

$$\|f - \mathrm{R}_\varrho(\Phi)\|_{L^2} \leq \varepsilon_k \leq \frac{\delta_{\varepsilon_k}}{2k} < \delta_{\varepsilon_k} = \min\{\|f - \mathrm{R}_\varrho(\Psi)\|_{L^2} \,:\, \Psi \in \mathcal{NN}^{\mathcal{B}}_{\Omega_{\varepsilon_k}, K_{\varepsilon_k}, d}\}.$$

But in case of $M \leq \Omega_{\varepsilon_k}$, we would have (as above) that $\Phi \in \mathcal{NN}^{\mathcal{B}}_{M, K_{\varepsilon_k}, d} \subset \mathcal{NN}^{\mathcal{B}}_{\Omega_{\varepsilon_k}, K_{\varepsilon_k}, d}$, which then yields a contradiction to the preceding inequality. Therefore, we must have

$$M > \Omega_{\varepsilon_k} = \left\lceil \varepsilon_k^{-2(d-1)/\beta} / [\log_2(1/\varepsilon_k) \cdot \log_2(\log_2(1/\varepsilon_k))] \right\rceil \geq \frac{\varepsilon_k^{-2(d-1)/\beta}}{\log_2(1/\varepsilon_k) \cdot \log_2(\log_2(1/\varepsilon_k))}.$$

Since $M \in \mathbb{N}$ was chosen arbitrarily, only subject to the restriction that there is $\Phi \in \mathcal{NN}^{\mathcal{B}}_{M, \lceil C_0 \log_2(1/\varepsilon_k) \rceil, d}$ with $\|f - \mathrm{R}_\varrho(\Phi)\|_{L^2} \leq \varepsilon_k$, this implies $M^{\mathcal{B}, \varrho, C_0}_{\varepsilon_k}(f) > \Omega_{\varepsilon_k}$, as claimed. □

C Depth matters: Fast approximation needs deep networks

In this section, we provide the proofs for the theorems from Subsection 4.2. In the whole section, ϱ will always be the ReLU function $\varrho(x) = \max\{0, x\}$, and all realizations are made using this activation function.

The overall proof strategy in this section is heavily inspired by Yarotsky [48]: At first, we exclusively work in dimension $d = 1$. For this setting, we begin by establishing (in Lemma C.1) a lower bound on the L^p approximation quality of affine-linear functions to the square function. By locally approximating a nonlinear C^3 function by its Taylor polynomial of degree two, this then implies (see Corollary C.3) a lower bound on the L^p approximation quality of affine-linear functions to nonlinear C^3 functions.

We then move to dimension $d > 1$ by saying that $g : \mathbb{R}^d \to \mathbb{R}$ is *P-piecewise slice affine* for some $P \in \mathbb{N}$ if each of the "slices" $t \mapsto g(x_0 + tv_0)$ for arbitrary $x_0, v_0 \in \mathbb{R}^d$ is piecewise affine-linear with at most P pieces. By applying a "Fubini-type argument", we lift the one-dimensional lower bounds to a lower bound for the L^p approximation quality that can be achieved for approximating a nonlinear, d-dimensional C^3 function using P-piecewise slice affine functions, see Proposition C.5.

We then complete the proof (see Theorem C.6) by invoking known results of Telgarsky [47] which show that realizations of ReLU neural networks are always P-piecewise slice affine, for $P \lesssim N^L$, where N is the number of neurons of the network, and L is its depth.

The main difference to the results by Yarotsky [48] is that Yarotsky considers approximation in L^∞, while we are interested in approximation in the L^p-sense, with $p < \infty$. In this case, the reduction of the d-dimensional case to the one-dimensional case is more involved; see the proof of Proposition C.5.

After this high-level overview, let us turn to the details:

Lemma C.1. *There is a universal constant $C_0 > 0$ with the following property: For arbitrary $\alpha, a, b \in \mathbb{R}$ with $a < b$ and $1 \leq p < \infty$, we have*

$$\inf_{\beta,\gamma\in\mathbb{R}} \left\|\alpha \cdot x^2 - (\beta x + \gamma)\right\|_{L^p([a,b];dx)} \geq C_0 \cdot |\alpha| \cdot (b-a)^{2+\frac{1}{p}} .$$

Proof. For $\alpha = 0$, the claim is trivial. Next, for $\alpha \neq 0$, we have

$$\begin{aligned}\inf_{\beta,\gamma\in\mathbb{R}} \left\|\alpha \cdot x^2 - (\beta x + \gamma)\right\|_{L^p([a,b];dx)} &= |\alpha| \cdot \inf_{\beta,\gamma\in\mathbb{R}} \left\|x^2 - \left(\frac{\beta}{\alpha}x + \frac{\gamma}{\alpha}\right)\right\|_{L^p([a,b];dx)} \\ &= |\alpha| \cdot \inf_{\beta',\gamma'\in\mathbb{R}} \left\|x^2 - (\beta' \cdot x + \gamma')\right\|_{L^p([a,b];dx)} .\end{aligned}$$

This easily shows that it suffices to consider the case $\alpha = 1$.

Next, let us consider the case $a = 0$ and $b = 1$. By Jensen's inequality, $\|f\|_{L^p([0,1])} \geq \|f\|_{L^1([0,1])}$ for $1 \leq p < \infty$, and $(b-a)^{2+\frac{1}{p}} = 1$ for arbitrary p, so that it suffices to consider the case $p = 1$. Next, consider the space $V := \operatorname{span}\{1, \mathrm{id}\} \leq L^1([0,1])$, which is finite dimensional, and hence closed. Since $(x \mapsto x^2) \notin V$, the absolute constant $C_0 := \operatorname{dist}\left((x \mapsto x^2), V\right)$ is positive. This proves the claim in case of $a = 0$ and $b = 1$.

Finally, for the general case, first note by a straightforward application of the change-of-variables formula that $\|f\|_{L^p([a,b])} = (b-a)^{1/p} \cdot \|f(a + (b-a)y)\|_{L^p([0,1];dy)}$ for measurable $f : [a,b] \to \mathbb{R}$. Applied to our specific setting, this implies for arbitrary $\beta, \gamma \in \mathbb{R}$ that

$$\begin{aligned}\left\|x^2 - (\beta x + \gamma)\right\|_{L^p([a,b];dx)} &= (b-a)^{\frac{1}{p}} \cdot \left\|(a + (b-a)y)^2 - [\beta(a + (b-a)y) + \gamma]\right\|_{L^p([0,1];dy)} \\ &= (b-a)^{2+\frac{1}{p}} \cdot \left\|y^2 - \left[\frac{\beta - 2a}{b-a}y + \frac{\beta a + \gamma - a^2}{(b-a)^2}\right]\right\|_{L^p([0,1];dy)} \geq C_0 \cdot (b-a)^{2+\frac{1}{p}} .\end{aligned}$$

As seen at the beginning of the proof, this yields the claim. ∎

The preceding lemma shows that affine-linear functions cannot approximate the square function too well. By approximating C^3 functions by their Taylor polynomial of degree 2, this implies that C^3 functions with nonvanishing second derivative are not approximated too well by linear functions. This is made precise by the following lemma:

Lemma C.2. *Let $f \in C^3([0,1])$ with $|f''(x)| \geq c > 0$ for all $x \in [0,1]$ and with $\|f'''\|_{\sup} \leq C$, for some $C > 0$. Then, with C_0 as in Lemma C.1, we have for arbitrary $1 \leq p < \infty$ that*

$$\inf_{\beta,\gamma\in\mathbb{R}} \|f(x) - (\beta x + \gamma)\|_{L^p([0,1];dx)} \geq \min\left\{\frac{C_0}{4} \cdot c, \frac{C_0^3}{8} \cdot \frac{c^3}{C^2}\right\} .$$

Proof. Set $N := \lceil 2/(3C_0) \cdot C/c \rceil \in \mathbb{N}$. For $i \in \underline{N+1}$ let $x_i := (i-1)/N \in [0,1]$. By Taylor's theorem, we know for each $i \in \underline{N}$ and $x \in (x_i, x_{i+1})$ that there is some $\xi_x \in (x_i, x) \subset (x_i, x_{i+1})$ with

$$\begin{aligned} f(x) &= f(x_i) + f'(x_i) \cdot (x - x_i) + \frac{f''(x_i)}{2} \cdot (x - x_i)^2 + \frac{f'''(\xi_x)}{6} \cdot (x - x_i)^3 \\ &= \frac{f''(x_i)}{2} \cdot x^2 + x \cdot [f'(x_i) - x_i \cdot f''(x_i)] + \left[f(x_i) - f'(x_i) \cdot x_i + \frac{1}{2} \cdot f''(x_i) \cdot x_i^2 \right] + \frac{f'''(\xi_x)}{6} \cdot (x - x_i)^3 \\ &=: \alpha_i \cdot x^2 + \beta_i \cdot x + \gamma_i + \frac{f'''(\xi_x)}{6} \cdot (x - x_i)^3 . \end{aligned}$$

Hence, since $|f'''(\xi_x)| \le C$, we get

$$\begin{aligned} \left\| f(x) - \left[\alpha_i \cdot x^2 + \beta_i \cdot x + \gamma_i \right] \right\|_{L^p([x_i, x_{i+1}]; dx)} &\le (x_{i+1} - x_i)^{\frac{1}{p}} \cdot \left\| f(x) - \left[\alpha_i \cdot x^2 + \beta_i \cdot x + \gamma_i \right] \right\|_{L^\infty([x_i, x_{i+1}]; dx)} \\ &\le N^{-\frac{1}{p}} \cdot \frac{C}{6} \cdot N^{-3} = \frac{C}{6} \cdot N^{-\left(3 + \frac{1}{p}\right)} . \end{aligned}$$

Therefore, by applying Lemma C.1 and by noting $|\alpha_i| = |f''(x_i)/2| \ge c/2$, we get for arbitrary $\beta, \gamma \in \mathbb{R}$ and $1 \le i \le N$ the estimate

$$\begin{aligned} &\| f(x) - (\beta x + \gamma) \|_{L^p([x_i, x_{i+1}]; dx)} \\ &\ge \left\| \alpha_i \cdot x^2 + \beta_i \cdot x + \gamma_i - (\beta x + \gamma) \right\|_{L^p([x_i, x_{i+1}]; dx)} - \left\| f(x) - \left[\alpha_i \cdot x^2 + \beta_i \cdot x + \gamma_i \right] \right\|_{L^p([x_i, x_{i+1}]; dx)} \\ &\ge \frac{c}{2} \cdot C_0 \cdot N^{-\left(2 + \frac{1}{p}\right)} - \frac{C}{6} \cdot N^{-\left(3 + \frac{1}{p}\right)} = \frac{c}{2} \cdot C_0 \cdot N^{-\left(2 + \frac{1}{p}\right)} \cdot \left(1 - \frac{1}{3C_0} \cdot \frac{C}{c} \cdot N^{-1} \right) . \end{aligned}$$

By our choice of N, we have $1/(3C_0) \cdot C/c \cdot N^{-1} \le 1/2$, so that $\| f(x) - (\beta x + \gamma) \|_{L^p([x_i, x_{i+1}]; dx)} \ge C_0/4 \cdot c \cdot N^{-\left(2 + \frac{1}{p}\right)}$. Hence,

$$\| f(x) - (\beta x + \gamma) \|_{L^p([0,1]; dx)} = \left[\sum_{i=1}^{N} \| f(x) - (\beta x + \gamma) \|_{L^p([x_i, x_{i+1}]; dx)}^p \right]^{\frac{1}{p}} \ge \frac{C_0}{4} \cdot c \cdot N^{-2}.$$

For brevity, set $\theta := 2/(3C_0) \cdot C/c$. There are now two cases: First, if $\theta < 1$, then $N = 1$, so that we get $\| f(x) - (\beta x + \gamma) \|_{L^p([0,1]; dx)} \ge C_0/4 \cdot c$, i.e., the claim is valid in this case. Finally, if $\theta \ge 1$, then $N = \lceil \theta \rceil \le 1 + \theta \le 2\theta$, and thus

$$\| f(x) - (\beta x + \gamma) \|_{L^p([0,1]; dx)} \ge \frac{C_0}{4} \cdot c \cdot N^{-2} \ge \frac{C_0}{4} \cdot c \cdot (2\theta)^{-2} = \frac{C_0}{16} \cdot c \cdot \left(\frac{3C_0}{2} \cdot \frac{c}{C} \right)^2 \ge \frac{C_0^3}{8} \cdot \frac{c^3}{C^2},$$

so that the claim also holds in this case. ∎

The next lemma generalizes the preceding estimate from the interval $[0,1]$ to general intervals $[a,b]$.

Corollary C.3. *Let $c, C > 0$ be arbitrary and let $C_0 > 0$ as in Lemma C.1. Further, let $a, b \in \mathbb{R}$ with $0 < b - a < \frac{1}{2} C_0 \cdot \frac{c}{C}$.*

Then, each function $f \in C^3([a,b])$ with $\|f'''\|_{\sup} \le C$ and with $|f''(x)| \ge c$ for all $x \in [a,b]$ satisfies

$$\inf_{\beta, \gamma \in \mathbb{R}} \| f(x) - (\beta x + \gamma) \|_{L^p([a,b]; dx)} \ge \frac{C_0}{4} \cdot c \cdot (b - a)^{2 + \frac{1}{p}} \qquad \forall\, 1 \le p < \infty.$$

Proof. Define

$$\widetilde{f} : [0,1] \to \mathbb{R}, x \mapsto f(a + (b-a)x),$$

and note $\widetilde{f} \in C^3([0,1])$ with $|\widetilde{f}''(x)| = (b-a)^2 \cdot |f''(a + (b-a)x)| \ge (b-a)^2 \cdot c =: c'$, as well as

$$\|\widetilde{f}'''\|_{\sup} = (b-a)^3 \cdot \|f'''\|_{\sup} \le (b-a)^3 \cdot C =: C'.$$

By our assumptions on a, b, c, C, we then have

$$\left(\frac{C_0^3}{8} \cdot \frac{(c')^3}{(C')^2} \right) \Big/ \left(\frac{C_0}{4} \cdot c' \right) = \frac{C_0^3}{8} \cdot \frac{(b-a)^6 \cdot c^3}{(b-a)^6 \cdot C^2} \cdot \frac{4}{C_0} \cdot \frac{1}{(b-a)^2 \cdot c} = \frac{C_0^2}{2} \cdot \frac{c^2}{C^2} \cdot \frac{1}{(b-a)^2} \ge \left(\frac{C_0}{2} \cdot \frac{c}{C} \cdot \frac{1}{b-a} \right)^2 > 1,$$

so that Lemma C.2 shows

$$\inf_{\beta, \gamma \in \mathbb{R}} \left\| \widetilde{f}(x) - (\beta x + \gamma) \right\|_{L^p([0,1]; dx)} \ge \min \left\{ \frac{C_0}{4} \cdot c', \frac{C_0^3}{8} \cdot \frac{(c')^3}{(C')^2} \right\} = \frac{C_0}{4} \cdot c' = \frac{C_0}{4} \cdot c \cdot (b-a)^2 .$$

To complete the proof, we note from a direct application of the change-of-variables formula for arbitrary $\beta, \gamma \in \mathbb{R}$ that

$$\|f(x) - (\beta x + \gamma)\|_{L^p([a,b];dx)} = (b-a)^{\frac{1}{p}} \cdot \left\| \widetilde{f}(y) - [\beta((b-a)y + a) + \gamma] \right\|_{L^p([0,1];dy)} \geq \frac{C_0}{4} \cdot c \cdot (b-a)^{2+\frac{1}{p}}. \qquad \square$$

Before we progress further, we introduce a convenient terminology:

Definition C.4. *Let $P \in \mathbb{N}$. A function $g : \mathbb{R}^d \to \mathbb{R}$ is called **P-piecewise slice affine** if for arbitrary $x_0, v \in \mathbb{R}^d$ the function $g_{x_0,v} : \mathbb{R} \to \mathbb{R}, t \mapsto g(x_0 + tv)$ is piecewise affine-linear with at most P pieces. Precisely, this means that there are $-\infty = t_0 < t_1 < \cdots < t_P = \infty$ such that $g_{x_0,v}|_{(t_i,t_{i+1})}$ is affine-linear for each $i \in \{0, \dots, P-1\}$.*

Remark. *Note that we allow $g_{x_0,v}$ to even be discontinuous at the "break points" $t_1, \dots, t_{P-1}$.*

Our next result shows that if a P-piecewise slice affine function approximates a nonlinear function $f \in C^3(\Omega)$ very well, then P needs to be large. This result will then imply that ReLU networks need to have a certain minimal depth in order to achieve a given approximation rate for nonlinear functions, once we show that if $g = R_\varrho(\Phi)$, then g is P-piecewise slice affine for $P \asymp [N(\Phi)]^{L(\Phi)}$. Actually, we will not derive this claim from first principles, but rather use existing results of Telgarsky [47]. But first, let us consider the case of general P-piecewise slice affine functions:

Proposition C.5. *Let $\Omega \subset \mathbb{R}^d$ be nonempty, open, bounded and connected, and let $f \in C^3(\Omega)$ be nonlinear, i.e., there do* not *exist $y_0 \in \mathbb{R}$ and $w \in \mathbb{R}^d$ with $f(x) = y_0 + \langle w, x\rangle$ for all $x \in \Omega$. Then there is a constant $C_f > 0$ with the following property:*

If $g : \mathbb{R}^d \to \mathbb{R}$ is measurable and P-piecewise slice affine for some $P \in \mathbb{N}$, then we have

$$\|f - g\|_{L^p(\Omega)} \geq C_f \cdot P^{-\left(2+\frac{1}{p}\right)} \qquad \text{for all } 1 \leq p < \infty.$$

Proof. Let $\operatorname{Hess} f = D(\nabla f)$ denote the Hessian of f. If we had $\operatorname{Hess} f \equiv 0$, then it would follow by standard results of multivariable calculus (since Ω is connected) that ∇f is constant, and then that $f(x) = f(x_0) + \langle \nabla f(x_0), x - x_0\rangle$ for all $x \in \Omega$, where $x_0 \in \Omega$ is fixed, but arbitrary. Since f is assumed nonlinear, this is impossible.

Hence, let $x_0 \in \Omega$ with $\operatorname{Hess} f(x_0) \neq 0$. Since $A := \operatorname{Hess} f(x_0)$ is symmetric, the spectral theorem shows that there is an orthonormal basis $(b_1, \dots, b_d)$ of $\mathbb{R}^d$ that consists of eigenvectors for A, and at least one of these eigenvectors needs to correspond to a non-zero eigenvalue; by rearranging we can assume $Ab_d = \lambda \cdot b_d$ for some $\lambda \in \mathbb{R} \setminus \{0\}$. Since Ω is open, there is some $\varepsilon \in (0, 1/2)$ with $\overline{B_{d\varepsilon}}(x_0) \subset \Omega$. Since $|\langle \operatorname{Hess} f(x_0) b_d, b_d\rangle| = |\lambda| \neq 0$, and since $\operatorname{Hess} f$ is continuous, we can possibly shrink ε to achieve $|\langle \operatorname{Hess} f(x) b_d, b_d\rangle| \geq c := \frac{|\lambda|}{2}$ for all $x \in \overline{B_{d\varepsilon}}(x_0)$. Furthermore, since $\overline{B_{d\varepsilon}}(x_0) \subset \Omega$ is compact, the constant

$$C := d^3 \cdot \sup_{x \in \overline{B_{d\varepsilon}}(x_0)} \max_{|\alpha|=3} |\partial^\alpha f(x)|$$

is finite. Finally, by again shrinking ε (which can at most *shrink* C), we can assume $2\varepsilon < \frac{1}{2} C_0 \cdot \frac{c}{C}$, where C_0 is the constant from Lemma C.1.

Now, for $y = (y_1, \dots, y_{d-1}) \in \mathbb{R}^{d-1}$ let us set $z_y := x_0 + \sum_{i=1}^{d-1} y_i b_i$. Note $z_y + t \cdot b_d \in \overline{B_{d\varepsilon}}(x_0)$ for all $y \in [-\varepsilon, \varepsilon]^{d-1}$ and $t \in [-\varepsilon, \varepsilon]$. Therefore, since $(b_1, \dots, b_d)$ is an orthonormal basis, an application of the change-of-variables formula and of Fubini's theorem shows

$$\|f - g\|_{L^p(\Omega)}^p \geq \int_{B_{d\varepsilon}(x_0)} |f(x) - g(x)|^p \, dx \geq \int_{[-\varepsilon,\varepsilon]^{d-1}} \int_{-\varepsilon}^{\varepsilon} |f(z_y + t \cdot b_d) - g(z_y + t \cdot b_d)|^p \, dt \, dy.$$

Note that the choice of $x_0, \varepsilon, (b_1, \dots, b_d)$ and λ, c, C, C_0 are all independent of g and P.

Now, let $y \in [-\varepsilon, \varepsilon]^{d-1}$ be fixed, but arbitrary. Since g is P-piecewise slice affine, we know that the map $g_{z_y, b_d} : \mathbb{R} \to \mathbb{R}, t \mapsto g(z_y + t \cdot b_d)$ is piecewise affine-linear, with at most P pieces, i.e., there is a partition $\mathbb{R} = \biguplus_{i=1}^N I_i$ (up to a null-set) into open intervals $I_1, \dots, I_N$ with $N \in \underline{P}$ such that g_{z_y, b_d} is affine-linear on each I_i. Hence, with λ denoting the one-dimensional Lebesgue measure, we conclude $2\varepsilon = \lambda\left(\biguplus_{i=1}^N (I_i \cap [-\varepsilon, \varepsilon])\right) = \sum_{i=1}^N \lambda([-\varepsilon, \varepsilon] \cap I_i)$, so that $\lambda([-\varepsilon, \varepsilon] \cap I_i) \geq 2\varepsilon/N \geq 2\varepsilon/P$ for some $i \in \underline{N}$. Therefore, $[-\varepsilon, \varepsilon] \cap I_i \supset (a_y, b_y)$ for certain $a_y, b_y \in [-\varepsilon, \varepsilon]$ with $b_y - a_y \geq 2\varepsilon/P$. Since g_{z_y, b_d} is affine-linear on $(a_y, b_y) \subset I_i \cap [-\varepsilon, \varepsilon]$, there are thus certain $\beta_y, \gamma_y \in \mathbb{R}$ with

$$\int_{-\varepsilon}^{\varepsilon} |f(z_y + t \cdot b_d) - g(z_y + t \cdot b_d)|^p \, dt \geq \|f(z_y + t \cdot b_d) - (\beta_y t + \gamma_y)\|_{L^p([a_y, b_y]; dt)}^p.$$

But with

$$f_y : [a_y, b_y] \to \mathbb{R}, t \mapsto f(z_y + t \cdot b_d)$$

we have $f_y \in C^3([a_y, b_y])$ and $|f_y''(t)| = |\langle \mathrm{Hess}\, f(z_y + t \cdot b_d) \cdot b_d, b_d \rangle| \geq c$ for all $t \in [a_y, b_y] \subset [-\varepsilon, \varepsilon]$, since $z_y + t \cdot b_d \in \overline{B_{d\varepsilon}}(x_0)$, as we saw above. Finally, by an iterated application of the chain rule, we also have

$$|f_y'''(t)| = \left| \sum_{i,j,\ell=1}^{d} (b_d)_i (b_d)_j (b_d)_\ell \cdot (\partial_i \partial_j \partial_\ell f)(z_y + t \cdot b_d) \right| \leq d^3 \cdot \sup_{x \in \overline{B_{d\varepsilon}(x_0)}} \max_{|\alpha|=3} |\partial^\alpha f(x)| = C,$$

where we used that $|(b_d)_i| \leq |b_d| = 1$ for all $i \in \underline{d}$. All in all, an application of Corollary C.3 now shows because of $b_y - a_y \leq 2\varepsilon < {C_0}/{2} \cdot {c}/{C}$ that

$$\|f(z_y + t \cdot b_d) - (\beta_y t + \gamma_y)\|_{L^p([a_y,b_y];dt)}^p \geq \left[\frac{C_0}{4} \cdot c \cdot (b_y - a_y)^{2+\frac{1}{p}} \right]^p \geq \left(\frac{C_0}{4} \cdot c \right)^p \cdot \left(\frac{2\varepsilon}{P} \right)^{2p+1}.$$

By putting everything together, and by recalling $\varepsilon < 1/2$ and $p \geq 1$, so that $(2\varepsilon)^{2p+d} \geq (2\varepsilon)^{(d+2)\cdot p}$, we thus arrive at

$$\begin{aligned}
\|f - g\|_{L^p(\Omega)}^p &\geq \int_{[-\varepsilon,\varepsilon]^{d-1}} \int_{-\varepsilon}^{\varepsilon} |f(z_y + t \cdot b_d) - g(z_y + t \cdot b_d)|^p \, dt \, dy \\
&\geq \int_{[-\varepsilon,\varepsilon]^{d-1}} \left(\frac{C_0}{4} \cdot c \right)^p \cdot \left(\frac{2\varepsilon}{P} \right)^{2p+1} dy \geq (2\varepsilon)^{(d+2)p} \cdot \left(\frac{C_0}{4} \cdot c \right)^p \cdot P^{-(1+2p)},
\end{aligned}$$

which yields the claim if we set $C_f := (2\varepsilon)^{d+2} \cdot {C_0}/{4} \cdot c$. Note that $C_f > 0$ is indeed independent of g and P. □

By using the results of Telgarsky[47] which show that functions represented by neural ReLU networks are P-piecewise slice affine for $P \asymp [N(\Phi)]^{L(\Phi)}$, we can now derive a lower bound on the number of layers that are needed to achieve a given approximation rate for nonlinear C^3 functions:

Theorem C.6. *Let $\Omega \subset \mathbb{R}^d$ be nonempty, open, bounded, and connected. Furthermore, let $f \in C^3(\Omega)$ be nonlinear. Then there is a constant $C_f > 0$ satisfying*

$$\begin{aligned}
\|f - R_\varrho(\Phi)\|_{L^p(\Omega)} &\geq C_f \cdot (N(\Phi) - 1)^{-L(\Phi)\cdot\left(2+\frac{1}{p}\right)}, \\
\|f - R_\varrho(\Phi)\|_{L^p(\Omega)} &\geq C_f \cdot (M(\Phi) + d)^{-L(\Phi)\cdot\left(2+\frac{1}{p}\right)}
\end{aligned}$$

for all $1 \leq p < \infty$ and each ReLU neural network Φ with input dimension d and output dimension 1.

Remark. *By adapting the given arguments (mostly the proof of Lemma C.2), one can show that the same claim remains true for $f \in C^{2+\varepsilon}(\Omega)$, with fixed but arbitrary $\varepsilon > 0$. For the sake of brevity, we omitted this generalization.*

Before we give the proof of Theorem C.6, we observe the following corollary:

Corollary C.7. *Let $\Omega \subset \mathbb{R}^d$ be nonempty, open, bounded, and connected. Furthermore, let $f \in C^3(\Omega)$ be nonlinear. If there are constants $C, \theta > 0$, a null-sequence $(\varepsilon_k)_{k\in\mathbb{N}}$ of positive numbers, and a sequence $(\Phi_k)_{k\in\mathbb{N}}$ of ReLU neural networks satisfying*

$$\|f - R_\varrho(\Phi_k)\|_{L^p} \leq C \cdot \varepsilon_k \quad \text{and} \quad \left[M(\Phi_k) \leq C \cdot \varepsilon_k^{-\theta} \text{ or } N(\Phi_k) \leq C \cdot \varepsilon_k^{-\theta} \right]$$

for all $k \in \mathbb{N}$ and some $1 \leq p < \infty$, then

$$\liminf_{k\to\infty} L(\Phi_k) \geq \left(2 + \frac{1}{p} \right)^{-1} \cdot \frac{1}{\theta}.$$

Proof. Let us assume that the claim is false, i.e., we have $\liminf_{k\to\infty} L(\Phi_k) < (2 + 1/p)^{-1} \cdot 1/\theta$. By switching to a subsequence, we can then assume that there is some $\delta > 0$ with $L(\Phi_k) \leq (2 + 1/p)^{-1} \cdot 1/\theta - \delta =: L$ for all $k \in \mathbb{N}$. Note that $1 \leq L(\Phi_k) \leq L$.

Next, since $\varepsilon_k \to 0$, and since $f \neq 0$ (because f is nonlinear), we have $\|f - R_\varrho(\Phi_k)\|_{L^p} < \|f\|_{L^p}$ for k large enough (which we will assume in the following). In particular, $R_\varrho(\Phi_k) \neq 0$ and hence $M(\Phi_k) \geq 1$, so that $M(\Phi_k) + d \leq (d+1) \cdot M(\Phi_k)$.

Now, there are two cases for (large) $k \in \mathbb{N}$: If $M(\Phi_k) \leq C \cdot \varepsilon_k^{-\theta}$, then the second part of Theorem C.6 shows

$$\begin{aligned} C_f \cdot (1+d)^{-L\left(2+\frac{1}{p}\right)} C^{-L\left(2+\frac{1}{p}\right)} \cdot \varepsilon_k^{L\theta\left(2+\frac{1}{p}\right)} &\leq C_f \cdot (1+d)^{-L\left(2+\frac{1}{p}\right)} \cdot [M(\Phi_k)]^{-L\left(2+\frac{1}{p}\right)} \\ &\leq C_f \cdot (M(\Phi_k)+d)^{-L\left(2+\frac{1}{p}\right)} \\ &\leq C_f \cdot (M(\Phi_k)+d)^{-L(\Phi_k)\left(2+\frac{1}{p}\right)} \\ &\leq \|f - R_\varrho(\Phi_k)\|_{L^p} \leq C \cdot \varepsilon_k. \end{aligned} \tag{C.1}$$

If otherwise $N(\Phi_k) \leq C \cdot \varepsilon_k^{-\theta}$, then the first part of Theorem C.6 shows

$$\begin{aligned} C_f \cdot C^{-L\left(2+\frac{1}{p}\right)} \cdot \varepsilon_k^{L\theta\left(2+\frac{1}{p}\right)} &\leq C_f \cdot (N(\Phi_k))^{-L\left(2+\frac{1}{p}\right)} \leq C_f \cdot (N(\Phi_k))^{-L(\Phi_k)\left(2+\frac{1}{p}\right)} \\ &\leq C_f \cdot (N(\Phi_k)-1)^{-L(\Phi_k)\left(2+\frac{1}{p}\right)} \leq \|f - R_\varrho(\Phi_k)\|_{L^p} \leq C \cdot \varepsilon_k. \end{aligned} \tag{C.2}$$

At least one of the equations (C.1) or (C.2) holds for infinitely many $k \in \mathbb{N}$. Since $\varepsilon_k \to 0$ and $\varepsilon_k > 0$, this easily yields $L\theta(2 + 1/p) \geq 1$, and hence $(2 + 1/p)^{-1} \cdot \theta^{-1} - \delta = L \geq (2 + 1/p)^{-1} \cdot \theta^{-1}$, which is the desired contradiction. □

We close this section with the proof of Theorem C.6.

Proof of Theorem C.6. **Step 1**: In this step, we show[‡] that if Φ is a neural network of depth L and with N neurons, then $R_\varrho(\Phi)$ is P-piecewise slice affine with $P \leq (2/L)^L \cdot (N-1)^L$.

To this end, we first introduce some terminology: As in [47], let us call a *continuous* function $f : \mathbb{R} \to \mathbb{R}$ ***t*-sawtooth** (with $t \in \mathbb{N}$) if f is piecewise affine-linear with at most t pieces, i.e., there are $-\infty = x_0 < x_1 < \cdots < x_t = \infty$ such that $f|_{(x_{i-1}, x_i)}$ is affine-linear for each $i \in \underline{t}$. Note that there are no issues at the boundary points of the affine-linear "pieces", since (in slight contrast to [47]), we assume f to be continuous. Using this terminology, [47, Lemma 2.3] states that if $f, g : \mathbb{R} \to \mathbb{R}$ are k-sawtooth and ℓ-sawtooth, respectively, then $f + g$ is $k + \ell$-sawtooth, and $f \circ g$ is $k\ell$-sawtooth. Note that the ReLU ϱ is 2-sawtooth.

Now, let $\Phi = \left(\left(A^{(1)}, b^{(1)}\right), \ldots, \left(A^{(L)}, b^{(L)}\right)\right)$ be a neural network with d-dimensional input and one-dimensional input, with L layers and N neurons, i.e., we have $A^{(\ell)} \in \mathbb{R}^{N_\ell \times N_{\ell-1}}$ and $b^{(\ell)} \in \mathbb{R}^{N_\ell}$, where $N_0 = d$ and $N_L = 1$, and $N = \sum_{j=0}^{L} N_j$. Further, let $g := R_\varrho(\Phi)$ and let $x, v \in \mathbb{R}^d$ be arbitrary. We want to show that $g_{x,v} : \mathbb{R} \to \mathbb{R}, t \mapsto g(x + t \cdot v)$ is P-sawtooth, with $P \leq (2/L)^L \cdot (N-1)^L$. To see this, inductively define $g^{(0)}, g^{(1)}, \ldots, g^{(L)}$ as follows: $g^{(0)} : \mathbb{R} \to \mathbb{R}^d, x \mapsto x + t \cdot v$,

$$g^{(\ell+1)} : \mathbb{R} \to \mathbb{R}^{N_{\ell+1}}, t \mapsto \varrho\left(A^{(\ell+1)} \cdot g^{(\ell)}(t) + b^{(\ell+1)}\right) \qquad \text{for } 0 \leq \ell \leq L-2,$$

and $g^{(L)} : \mathbb{R} \to \mathbb{R}, t \mapsto A^{(L)} \cdot g^{(L-1)}(t) + b^{(L)}$. We clearly have $g^{(L)} = g_{x,v}$.

We will show by induction on $\ell \in \{0, \ldots, L\}$ that each component function $g_i^{(\ell)}$ for $i \in \underline{N_\ell}$ is M_ℓ-sawtooth, with $M_\ell := \prod_{j=0}^{\ell-1} 2N_j$, where (by the convention for empty products) $M_0 = 1$. Indeed, for $\ell = 0$, we have $g_i^{(0)}(t) = x_i + t \cdot v_i$, which is affine-linear. Hence, $g_i^{(0)}$ is M_0-sawtooth, since $M_0 = 1$. For the induction step, assume that all $g_i^{(\ell)}$, $i \in \underline{N_\ell}$ are M_ℓ-sawtooth for some $0 \leq \ell \leq L-1$. In case of $\ell = L-1$, let $\theta := \mathrm{id}_{\mathbb{R}}$, and otherwise let $\theta := \varrho$. In either case, we have that θ is 2-sawtooth, and

$$g_i^{(\ell+1)}(t) = \theta\left(b_i^{(\ell+1)} + \sum_{j=1}^{N_\ell} A_{i,j}^{(\ell+1)} \cdot g_j^{(\ell)}(t)\right) \qquad \text{for } t \in \mathbb{R} \quad \text{and } i \in \underline{N_{\ell+1}}.$$

But since each $g_j^{(\ell)}$ is M_ℓ-sawtooth, so is $t \mapsto A_{i,j}^{(\ell+1)} \cdot g_j^{(\ell)}(t)$, so that $t \mapsto \sum_{j=1}^{N_\ell} A_{i,j}^{(\ell+1)} \cdot g_j^{(\ell)}(t)$ is $(M_{\ell+1}/2)$-sawtooth, since $\sum_{j=1}^{N_\ell} M_\ell = N_\ell \cdot M_\ell = 1/2 M_{\ell+1}$. Thus, since θ is 2-sawtooth, $g_i^{(\ell+1)}$ is $M_{\ell+1}$-sawtooth, as claimed.

All in all, we have shown that $g_{x,v} = g^{(L)}$ is M_L-sawtooth, where $M_L = \prod_{j=0}^{L-1} 2N_j = 2^L \cdot \prod_{j=0}^{L-1} N_j$. Now, by concavity of the natural logarithm and because of $N_L = 1$, we have

$$\ln\left(\frac{N(\Phi)-1}{L}\right) = \ln\left(\frac{1}{L}\sum_{j=0}^{L-1} N_j\right) \geq \frac{1}{L}\sum_{j=0}^{L-1} \ln N_j = \frac{1}{L} \cdot \ln\left(\prod_{j=0}^{L-1} N_j\right),$$

[‡]Essentially, this is already contained in the statement of [47, Lemma 2.1], but Telgarsky uses a slightly different definition of neural networks than we do. Therefore, and for the convenience of the reader, we provide a proof.

and hence $\prod_{j=0}^{L-1} N_j \leq e^{L\cdot \ln((N(\Phi)-1)/L)} = \left({}^{(N(\Phi)-1)}/_{L}\right)^L$, so that all in all $M_L \leq ({}^2/_L)^L \cdot (N(\Phi)-1)^L$.

Step 2: Now, an application of Proposition C.5 yields a constant $C_f^{(0)} > 0$ (independent of Φ, L, p) satisfying

$$\|f - R_\varrho(\Phi)\|_{L^p} \geq C_f^{(0)} \cdot M_L^{-\left(2+\frac{1}{p}\right)} \geq C_f^{(0)} \cdot \left(\frac{L}{2}\right)^{L\left(2+\frac{1}{p}\right)} \cdot (N(\Phi)-1)^{-L\left(2+\frac{1}{p}\right)} \geq \frac{1}{8} \cdot C_f^{(0)} \cdot (N(\Phi)-1)^{-L\left(2+\frac{1}{p}\right)}.$$

Here, the estimate $({}^L/_2)^{L(2+{}^1/_p)} \geq {}^1/_8$ can be easily seen to be true by distinguishing the cases $L = 1$ and $L \geq 2$, and by noting $2 \leq 2 + {}^1/_p \leq 3$. This yields the first claim, with $C_f = C_f^{(0)}/8$, since $L = L(\Phi)$.

Step 3: Finally, to prove the second claim, recall from Lemma E.1 that there is a neural network Φ' with $R_\varrho(\Phi) = R_\varrho(\Phi')$ and such that $M(\Phi') \leq M(\Phi)$ and $N(\Phi') \leq M(\Phi') + d + 1$, as well as $L(\Phi') \leq L(\Phi)$. By applying the first claim of the current theorem to Φ' instead of Φ (and with $L' = L(\Phi')$ instead of $L = L(\Phi)$), we get

$$\begin{aligned}\|f - R_\varrho(\Phi)\|_{L^p} = \|f - R_\varrho(\Phi')\|_{L^p} &\geq C_f \cdot (N(\Phi')-1)^{-L'\left(2+\frac{1}{p}\right)} \geq C_f \cdot (N(\Phi')-1)^{-L\left(2+\frac{1}{p}\right)} \\ &\geq C_f \cdot (M(\Phi')+d)^{-L\left(2+\frac{1}{p}\right)} \geq C_f \cdot (M(\Phi)+d)^{-L\cdot\left(2+\frac{1}{p}\right)}.\end{aligned}$$ □

D An estimate of intermediate derivatives

Lemma D.1. *For $d \in \mathbb{N}$, $\sigma \in (0,1]$ and $f \in C([0,1]^d)$, define*

$$\mathrm{Lip}_\sigma(f) := \sup_{\substack{x,y\in[0,1]^d \\ x\neq y}} \frac{|f(x)-f(y)|}{|x-y|^\sigma} \in [0,\infty].$$

Then, for $n \in \mathbb{N}_0$, $d \in \mathbb{N}$ and $\sigma \in (0,1]$ there is a constant $C = C(n,d,\sigma) > 0$ such that every $f \in C^n([0,1]^d)$ satisfies

$$\|\partial^\gamma f\|_{\sup} \leq C \cdot \left(\|f\|_{\sup} + \sum_{|\alpha|=n} \mathrm{Lip}_\sigma(\partial^\alpha f)\right) \qquad \textit{for all } \gamma \in \mathbb{N}_0^d \textit{ with } |\gamma| \leq n.$$

Proof. Note: This proof is heavily based on that of [1, Lemmas 4.10 and 4.12], where a related, but different estimate is established.

Step 1: We claim for $f \in C^1([0,1]^d)$ and arbitrary $N \in \mathbb{N}$ that

$$\|\partial_\ell f\|_{\sup} \leq 4 \cdot N^{\frac{1}{\sigma}} \cdot \|f\|_{\sup} + \frac{1}{N} \cdot \mathrm{Lip}_\sigma(\partial_\ell f) \qquad \text{for all } \ell \in \{1,\dots,d\}. \tag{D.1}$$

By symmetry (i.e., by relabeling the coordinates), we can assume $\ell = 1$. Define $K := \lceil N^{1/\sigma} \rceil$, and let $x = (x_1,\dots,x_d) \in [0,1]^d$ be arbitrary. Choose $i \in \{0,\dots,K-1\}$ with $x_1 \in [{}^i/_K, {}^{(i+1)}/_K]$. By the mean value theorem, there is some $\xi \in ({}^i/_K, {}^{(i+1)}/_K)$ with

$$|\partial_1 f(\xi, x_2,\dots,x_d)| = \left| \frac{f(\frac{i+1}{K}, x_2,\dots,x_d) - f(\frac{i}{K}, x_2,\dots,x_d)}{\frac{i+1}{K} - \frac{i}{K}} \right| \leq 2K \cdot \|f\|_{\sup} \leq 4 \cdot N^{\frac{1}{\sigma}} \cdot \|f\|_{\sup},$$

where we used $K = \lceil N^{1/\sigma} \rceil \leq 1 + N^{1/\sigma} \leq 2 \cdot N^{1/\sigma}$. Since $|(\xi, x_2,\dots,x_d) - x| \leq |\xi - x_1| \leq K^{-1} \leq N^{-1/\sigma}$, the preceding estimate implies

$$\begin{aligned}|\partial_1 f(x)| &\leq |\partial_1 f(x) - \partial_1 f(\xi, x_2,\dots,x_d)| + |\partial_1 f(\xi, x_2,\dots,x_d)| \\ &\leq (N^{-\frac{1}{\sigma}})^\sigma \cdot \mathrm{Lip}_\sigma(\partial_1 f) + 4 \cdot N^{\frac{1}{\sigma}} \cdot \|f\|_{\sup} = 4 \cdot N^{\frac{1}{\sigma}} \cdot \|f\|_{\sup} + \frac{1}{N} \cdot \mathrm{Lip}_\sigma(\partial_1 f),\end{aligned}$$

as claimed (since we assumed $\ell = 1$).

Step 2: For brevity, set $|f|_\ell := \sum_{|\alpha|=\ell} \|\partial^\alpha f\|_{\sup}$ and $|f|_{\ell,\sigma} := \sum_{|\alpha|=\ell} \mathrm{Lip}_\sigma(\partial^\alpha f) \in [0,\infty]$ for $\ell \in \mathbb{N}_0$ and $f \in C^\ell([0,1]^d)$. In this step, we show by induction on $k \in \mathbb{N}_0$ that for each $k \in \mathbb{N}_0$ and $N \in \mathbb{N}$, there is a constant $C_{\sigma,d,k,N} > 0$ with

$$|f|_k \leq \frac{1}{N} \cdot |f|_{k,\sigma} + C_{\sigma,d,k,N} \cdot \|f\|_{\sup} \qquad \text{for all } f \in C^k([0,1]^d). \tag{D.2}$$

Before we begin with the induction, we first show the following estimate:

$$|f|_{k,\sigma} \leq d^2 \cdot |f|_{k+1} \qquad \text{for all } k \in \mathbb{N}_0 \text{ and } f \in C^{k+1}([0,1]^d). \tag{D.3}$$

To prove Equation (D.3), first note because of $\mathrm{diam}([0,1]^d) = \sqrt{d}$ that each Lipschitz continuous function $f \in C([0,1]^d)$ satisfies $|f(x)-f(y)| \leq |x-y|^\sigma \cdot |x-y|^{1-\sigma} \cdot \mathrm{Lip}_1(f) \leq |x-y|^\sigma \cdot d^{(1-\sigma)/2} \cdot \mathrm{Lip}_1(f)$. Therefore, each $f \in C^1([0,1]^d)$ fulfills $\mathrm{Lip}_\sigma(f) \leq d^{(1-\sigma)/2} \cdot \mathrm{Lip}_1(f) \leq d^{(1-\sigma)/2} \cdot \|\nabla f\|_{\sup} \leq d \cdot \sum_{\ell=1}^d \|\partial_\ell f\|_{\sup}$, which finally yields for $f \in C^{k+1}([0,1]^d)$ that

$$|f|_{k,\sigma} = \sum_{|\alpha|=k} \mathrm{Lip}_\sigma(\partial^\alpha f) \leq d \sum_{\ell=1}^d \sum_{|\alpha|=k} \|\partial_\ell \partial^\alpha f\|_{\sup} \leq d^2 \cdot |f|_{k+1},$$

which is nothing but (D.3).

Now we properly begin with the proof of Equation (D.2). For $k=0$, Equation (D.2) is trivial with $C_{\sigma,d,0,N} = 1$, since $|f|_0 = \|f\|_{\sup}$. For $k=1$, Equation (D.2) is a consequence of Equation (D.1), which yields

$$\begin{aligned}|f|_k = |f|_1 = \sum_{\ell=1}^d \|\partial_\ell f\|_{\sup} &\leq 4 \cdot N^{\frac{1}{\sigma}} \cdot \|f\|_{\sup} \cdot \sum_{\ell=1}^d 1 + \frac{1}{N} \sum_{\ell=1}^d \mathrm{Lip}_\sigma(\partial_\ell f) \\ &= \frac{1}{N} \cdot |f|_{1,\sigma} + 4d \cdot N^{\frac{1}{\sigma}} \cdot \|f\|_{\sup} = \frac{1}{N} \cdot |f|_{k,\sigma} + 4d \cdot N^{\frac{1}{\sigma}} \cdot \|f\|_{\sup},\end{aligned}$$

so that $C_{\sigma,d,1,N} = 4d \cdot N^{1/\sigma}$ makes Equation (D.2) true for $k=1$.

For the induction step, note that if $f \in C^{k+1}([0,1]^d)$, and if we apply the case $k=1$ (with M instead of N) to each of the partial derivatives $\partial^\alpha f$ with $|\alpha| = k$, then we get

$$\begin{aligned}|f|_{k+1} &\leq \sum_{|\alpha|=k} |\partial^\alpha f|_1 \leq \sum_{|\alpha|=k} \left(\frac{1}{M} |\partial^\alpha f|_{1,\sigma} + C_{\sigma,d,M} \cdot \|\partial^\alpha f\|_{\sup} \right) \\ &\overset{(*)}{\leq} \frac{d^k}{M} \cdot |f|_{k+1,\sigma} + C_{\sigma,d,M} \cdot |f|_k \\ (\text{by induction}) &\leq \frac{d^k}{M} \cdot |f|_{k+1,\sigma} + C_{\sigma,d,M} \cdot \left(\frac{1}{N} \cdot |f|_{k,\sigma} + C_{\sigma,d,k,N} \cdot \|f\|_{\sup} \right) \\ (\text{by Eq. (D.3) since } f \in C^{k+1}([0,1]^d)) &\leq \frac{d^k}{M} \cdot |f|_{k+1,\sigma} + C_{\sigma,d,M} \cdot \left(\frac{d^2}{N} \cdot |f|_{k+1} + C_{\sigma,d,k,N} \cdot \|f\|_{\sup} \right),\end{aligned} \tag{D.4}$$

where $M, N \in \mathbb{N}$ can be chosen arbitrarily. In the above calculation, the step marked with $(*)$ used the elementary estimates $|\partial^\alpha f|_{1,\sigma} = \sum_{\ell=1}^d \mathrm{Lip}_\sigma(\partial_\ell \partial^\alpha f) \leq \sum_{|\gamma|=k+1} \mathrm{Lip}_\sigma(\partial^\gamma f) = |f|_{k+1,\sigma}$, which is valid for all $\alpha \in \mathbb{N}_0^d$ with $|\alpha| = k$, and $|\{\alpha \in \mathbb{N}_0^d : |\alpha| = k\}| \leq d^k$.

Finally, note that (D.2) is trivially satisfied (for $k+1$ instead of k) if $|f|_{k+1,\sigma} = \infty$. Therefore, we can assume $|f|_{k+1,\sigma} < \infty$. If we now choose $N = N(\sigma, d, M) \in \mathbb{N}$ to satisfy $N \geq 1 + 2d^2 C_{\sigma,d,M}$, so that $C_{\sigma,d,M} \cdot \frac{d^2}{N} \leq \frac{1}{2}$, then we get from Equation (D.4) by rearranging that

$$|f|_{k+1} \leq 2 \cdot \left(\frac{d^k}{M} \cdot |f|_{k+1,\sigma} + C_{\sigma,d,M} C_{\sigma,d,k,N} \cdot \|f\|_{\sup} \right) \leq \frac{2d^k}{M} \cdot |f|_{k+1,\sigma} + C'_{\sigma,d,k,M} \cdot \|f\|_{\sup}.$$

Since $M \in \mathbb{N}$ can be chosen arbitrarily, this establishes Equation (D.2) for $k+1$ instead of k, and thus completes the induction.

Step 3: For arbitrary $k \in \mathbb{N}$, we prove by induction on $0 \leq j \leq k-1$ that there is a constant $C_{\sigma,d,k,j} > 0$ with

$$|f|_{k-j} \leq |f|_{k,\sigma} + C_{\sigma,d,k,j} \cdot \|f\|_{\sup} \qquad \text{for all } f \in C^k([0,1]^d). \tag{D.5}$$

For $j = 0$, this is a direct consequence of Equation (D.2) (with $N = 1$). For the induction step, assume that (D.5) holds for some $0 \leq j \leq k-2$, and note

$$\begin{aligned}|f|_{k-(j+1)} &= |f|_{k-j-1} \\ (\text{Eq. (D.2) with } k-j-1 \text{ instead of } k \text{ and with } N = d^2) &\leq \frac{1}{d^2} \cdot |f|_{k-j-1,\sigma} + C'_{\sigma,d,k,j} \cdot \|f\|_{\sup} \\ (\text{Eq. (D.3) since } f \in C^k \subset C^{(k-j-1)+1}) &\leq |f|_{k-j} + C'_{\sigma,d,k,j} \cdot \|f\|_{\sup} \\ (\text{by induction}) &\leq |f|_{k,\sigma} + (C_{\sigma,d,k,j} + C'_{\sigma,d,k,j}) \cdot \|f\|_{\sup}.\end{aligned}$$

Step 4: In this step, we prove the actual claim. For $n = 0$, this is trivial, so that we can assume $n \geq 1$. Thus, let $f \in C^n([0,1]^d)$, and let $\gamma \in \mathbb{N}_0^d$ with $|\gamma| \leq n$. For $\gamma = 0$, the claim is trivial, so that we can assume $1 \leq |\gamma| \leq n$. Hence, $j := n - |\gamma|$ satisfies $0 \leq j \leq n-1$. Therefore, we can apply Step 3 with $k = n$ to conclude

$$\|\partial^\gamma f\|_{\sup} \leq |f|_{|\gamma|} = |f|_{n-j} \leq |f|_{n,\sigma} + C_{\sigma,d,n,j} \cdot \|f\|_{\sup}.$$

This easily implies the claim, with $C = \max\{1, \max\{C_{\sigma,d,n,j} : 0 \leq j \leq n-1\}\}$. □

E Reducing the number of neurons

In this short technical appendix, we prove that for each neural network Φ with one-dimensional output and d-dimensional input, one can assume essentially without loss of generality that $N(\Phi) \leq M(\Phi)+d+1$. This observation is important for the proof of Lemma B.4, where we encode the functions represented by a class of neural networks using a fixed number of bits. It is also used in the proof of Theorem C.6.

Lemma E.1. *Let $\varrho : \mathbb{R} \to \mathbb{R}$ with $\varrho(0) = 0$. Then, for every neural network Φ with input dimension $d \in \mathbb{N}$ and output dimension 1, there is a neural network Φ' with the same input and output dimension and with the following additional properties:*

- *We have $\mathrm{R}_\varrho(\Phi') = \mathrm{R}_\varrho(\Phi)$.*
- *We have $N(\Phi') \leq M(\Phi') + d + 1$.*
- *We have $M(\Phi') \leq M(\Phi)$ and $L(\Phi') \leq L(\Phi)$.*
- *If $I \subset \mathbb{R}$ contains the values of all non-zero weights of Φ, then the same holds for Φ'.*

Proof. Assume that $N(\Phi) > M(\Phi) + d + 1$, otherwise Φ itself admits all properties of the statement of the lemma. We show that in this case one can always find a network Φ' with $N(\Phi') < N(\Phi)$ and such that Φ' has the same input and output dimension as Φ, such that $M(\Phi') \leq M(\Phi)$, $L(\Phi') \leq L(\Phi)$ and $\mathrm{R}_\varrho(\Phi') = \mathrm{R}_\varrho(\Phi)$, and such that if $I \subset \mathbb{R}$ contains the values of all non-zero weights of Φ, then the same holds for Φ'. Iterating this observation yields the result.

For $n_1, n_2 \in \mathbb{N}$ and $A \in \mathbb{R}^{n_1 \times n_2}$, as well as $i \in \{1, \ldots, n_1\}$ we denote (in case of $n_1 > 1$) by $A_{\hat{i}} \in \mathbb{R}^{(n_1-1)\times n_2}$ the matrix resulting from removing the i-th row of A. Likewise, for $i \in \{1, \ldots, n_2\}$ we write (in case of $n_2 > 1$) $A^{\hat{i}}$ for the matrix resulting from removing the i-th column of A. Similarly, for $b \in \mathbb{R}^{n_1}$ with $n_1 > 1$, we denote by $b_{\hat{i}} \in \mathbb{R}^{n_1-1}$ the vector resulting from removing the i-th entry of b.

Let $\Phi = ((A_1, b_1), \ldots, (A_L, b_L))$ with $A_\ell \in \mathbb{R}^{N_\ell \times N_{\ell-1}}$ and $b_\ell \in \mathbb{R}^{N_\ell}$ for $\ell \in \{1, \ldots, L\}$. Since

$$\sum_{\ell=1}^{L} N_\ell - 1 = N(\Phi) - d - 1 > M(\Phi) = \sum_{\ell=1}^{L} \left(\|A_\ell\|_{\ell^0} + \|b_\ell\|_{\ell^0}\right),$$

there exist more rows of $[A_1, b_1], \ldots, [A_L, b_L]$ than non-zero entries in all these matrices. Hence, there exists $\ell \in \{1, \ldots, L\}$ and $i \in \{1, \ldots, N_\ell\}$ such that the i-th row of A_ℓ and the i-th entry of b_ℓ vanish. In fact, let us choose $\ell \in \{1, \ldots, L\}$ maximal with the property that there is some $i \in \{1, \ldots, N_\ell\}$ such that the i-th row of A_ℓ and the i-th entry of b_ℓ vanish. Now we distinguish three cases:

Case 1: If $N_\ell > 1$ (so that in particular $\ell < L$, since $N_L = 1$), then we set

$$\Phi' := ((A_1, b_1), \ldots, (A_{\ell-1}, b_{\ell-1}), ((A_\ell)_{\hat{i}}, (b_\ell)_{\hat{i}}), ((A_{\ell+1})^{\hat{i}}, b_{\ell+1}), (A_{\ell+2}, b_{\ell+2}), \ldots, (A_L, b_L)).$$

We have that $(A_{\ell+1})^{\hat{i}} x_{\hat{i}} = A_{\ell+1} x$ for all $x = (x_1, \ldots, x_{N_\ell}) \in \mathbb{R}^{N_\ell}$ with $x_i = 0$, and furthermore $(\varrho(A_\ell x + b_\ell))_{\hat{i}} = \varrho((A_\ell)_{\hat{i}}\, x + (b_\ell)_{\hat{i}})$ for all $x \in \mathbb{R}^{N_{\ell-1}}$. Since $\varrho(0) = 0$ we see that the i-th entry of $\varrho(A_\ell x + b_\ell)$ is zero, for arbitrary $x \in \mathbb{R}^{N_{\ell-1}}$. All in all, these observations show $\mathrm{R}_\varrho(\Phi') = \mathrm{R}_\varrho(\Phi)$. Moreover, $N(\Phi') < N(\Phi)$, $M(\Phi') \leq M(\Phi)$, and $L(\Phi') = L(\Phi)$ follow from the construction. The statement regarding the values of the nonzero weights being contained in I is also clearly satisfied.

Case 2: If $N_\ell = 1$, but $\ell > 1$, then we have $A_\ell = 0$ and $b_\ell = 0$. We set $\tilde{A}_1 := 0 \in \mathbb{R}^{1\times d}, \tilde{b}_1 := 0 \in \mathbb{R}$.

If $\ell < L$ we set

$$\Phi' := ((\tilde{A}_1, \tilde{b}_1), (A_{\ell+1}, b_{\ell+1}), \ldots, (A_L, b_L)).$$

By construction and because of $\varrho(0) = 0$, we have $\mathrm{R}_\varrho(\Phi') = \mathrm{R}_\varrho(\Phi)$ and $N(\Phi') < N(\Phi)$, as well as $M(\Phi') \leq M(\Phi)$ and $L(\Phi') \leq L(\Phi)$. The statement regarding the values of the nonzero weights being contained in I is also clearly satisfied.

If $\ell = L$, then $\mathrm{R}_\varrho(\Phi) = 0$. In this case, set

$$\Phi' := ((\tilde{A}_1, \tilde{b}_1)).$$

We then have $N(\Phi') = d+1 \leq M(\Phi)+d+1 < N(\Phi)$, as well as $M(\Phi') = 0 \leq M(\Phi)$ and $L(\Phi') = 1 \leq L(\Phi)$. Finally, since Φ' only has weights with value zero, the statement regarding the values of the nonzero weights being contained in I is trivially satisfied.

Case 3: If $\ell = 1$ and $N_1 = 1$, then $A_1 = b_1 = 0$. Thus we have

$$\sum_{\ell=2}^{L} N_\ell = \sum_{\ell=1}^{L} N_\ell - 1 = N(\Phi) - d - 1 > M(\Phi) = \sum_{\ell=2}^{L} (\|A_\ell\|_{\ell^0} + \|b_\ell\|_{\ell^0}),$$

and therefore there exists some $\ell' \in \{2, \ldots, L\}$ and some $j \in \{1, \ldots, N_{\ell'}\}$ such that the j-th row of $A_{\ell'}$ and the j-th entry of $b_{\ell'}$ vanish. This contradicts the maximality of ℓ, so that this case cannot occur. □

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